MATH PROBE

2

BYE ELLIOTT

Math Probe

2

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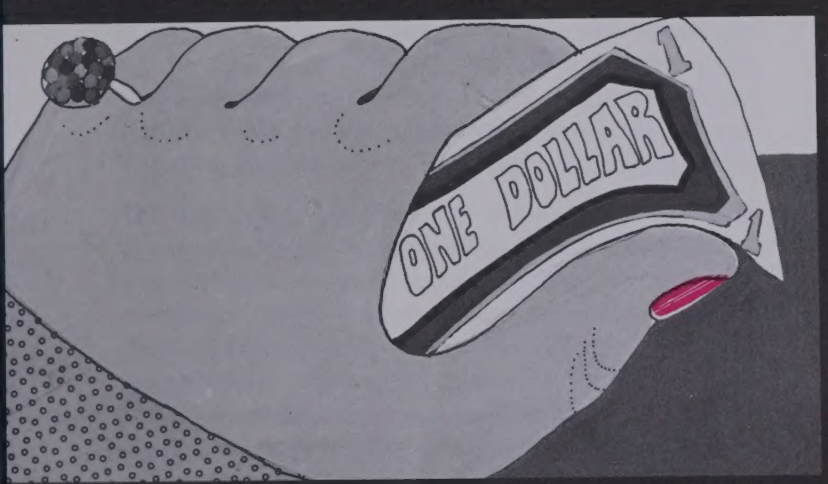

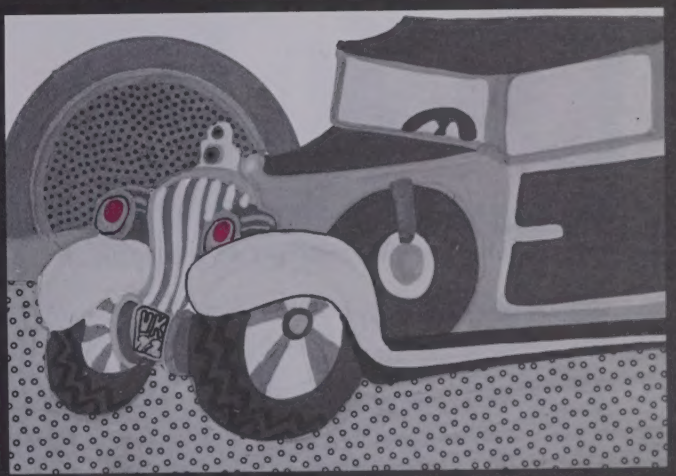

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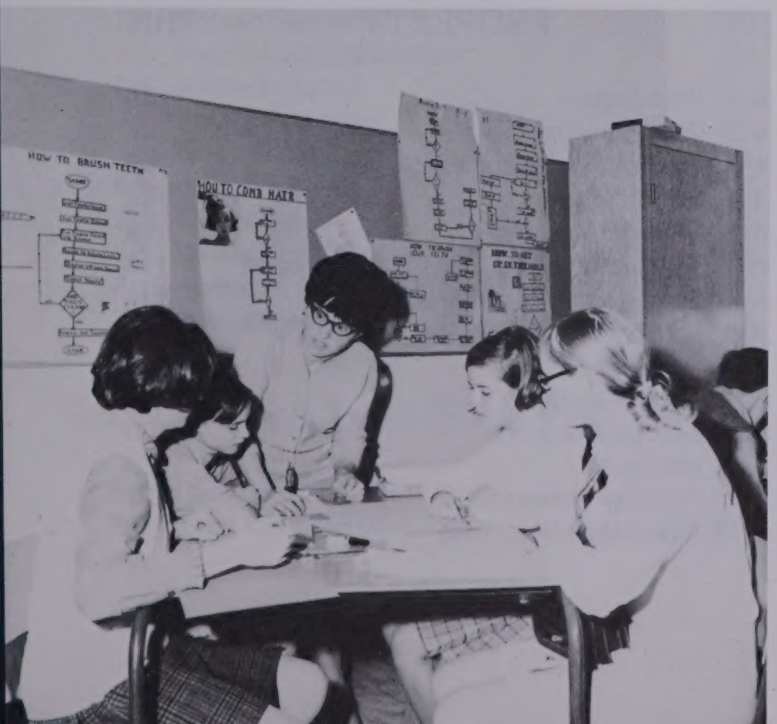
A Note From the Authors

To The Student:

This book is written to you—the student. A number of questions are asked just as if we were talking to you—and often no answers are given—immediately. You should think about the questions, try to answer them, discuss them, then read on.

There are several features of note in this book. It is intended that the Projects be done by groups of two to five students working cooperatively, each doing his part, then sharing the results. In most instances the Projects are to provide experiences upon which to build the formal concepts later. In some cases you will be able to do the exercises without further help. One important point: you can learn much by discussing the Projects with each other and your teacher. The discussions are a vital part of this program.

The Discovery Exercises are designed primarily for you to work by yourself though you may wish to check your conclusions with your classmates. While mathematics at times is something you must do alone, there are times when it is more efficient, more interesting, and more fun to work with others. But there are times when you must practise alone. Mathematics is like most sports—you only become good at it when you practise.



No book is complete within itself. This is true of this book. In a number of instances it is suggested the class write letters to obtain information, booklets, and publications. This should be done ahead of time to ensure the material is on hand when it is needed. Also it is suggested you invite people from your community into your classroom to discuss certain topics. At other times it is suggested that you visit the community and business world. In these ways mathematics is interwoven with the community and with meaningful aspects of living—it becomes relevant.

"Using the Library" is included in each chapter to encourage you to look at the history of mathematics, at some interesting applications, and at some of the interesting people who contributed to mathematics. Suggestion: look ahead—start the assignment early. Don't wait until the end of the chapter to start using the Library.

We hope you will enjoy probing mathematics.

M.P.B.

H.A.E.

2515239

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Chapter One
COMPUTATION

1·1 Using Mathematics

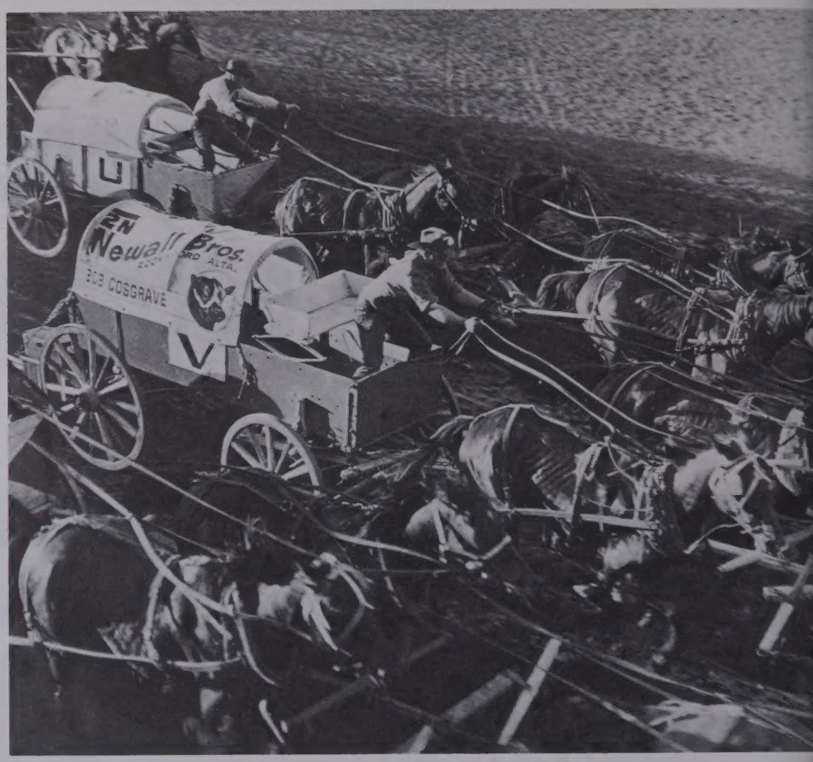
The attendance at the Calgary Stampede in 1964 was 573,562. The net revenue was $2,703,456. The attendance is given correct to the person. The attendance rounded to the nearest ten persons is 573,560. The chart shows this number rounded off to the various units.

The number	rounded to the nearest	is
573,562	ten	573,560
	hundred	573,600
	thousand	574,000
	ten thousand	570,000
	hundred thousand	600,000
	million	1,000,000

The same rules are used when rounding decimal fractions except the zeros are not attached. When the zeros are attached it indicates the place is being held.

The number	rounded to the nearest	is
4.3026	thousandth	4.303
	hundredth	4.30
	tenth	4.3

Note the zero here!



EXERCISE 1·1

1. After studying the two charts above, write the rules for rounding off numbers.

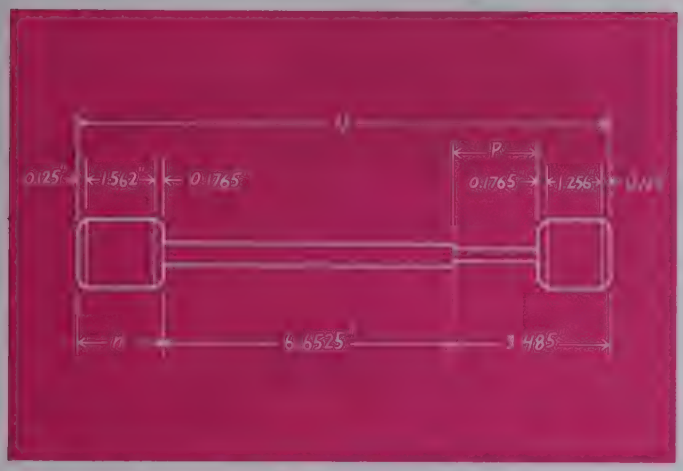
2. Use your rules to round off $2,703,456 to the nearest
   (a) ten dollars
   (b) hundred dollars
   (c) thousand dollars
   (d) ten thousand dollars
   (e) hundred thousand dollars
   (f) million dollars

   Hint! Remember to always work with the original number—not the previously rounded-off number.

3. Divide $2,703,456 by 573,562 to obtain the net revenue per person attending the Stampede.

4. Round off each number to the nearest hundredth:
   4.546 0.3764 7.499 41.455 3.749.

5. A machinist was turning this tool.



(a) Find the measurements of m, n, p correct to the nearest thousandth.

(b) The machinist was told that all measurements could be rounded off to the nearest hundredth. Draw the diagram and place on it all the measurements correct to the nearest hundredth.

(c) Find the measurements of m, n, p using the rounded off numbers in (b).

6. What is the difference between a measurement recorded as 23.4 m and 23.40 m?

7. The measurement 3.4 correct to the nearest tenth could actually be any of these (as well as others):

3.35, 3.36, 3.37, . . . ,
3.39, 3.40, 3.41, . . . , 3.44, 3.449,

but cannot be 3.45.

We write $3.35 \leq 3.4 < 3.45$ (1)

which is read *3.4 is equal to or greater than 3.35 but less than 3.45.*

Why do we say "3.4 is *equal to or greater than* 3.35" but not "*equal to or less than* 3.45"?

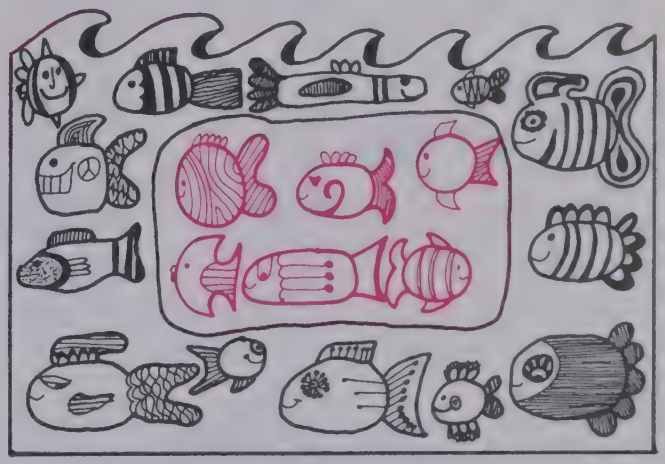
Write a range phrase like the statement in color (1) for each of these numbers which have been rounded to the nearest tenth.

4.6 5.3 14.8 5.2 8.9

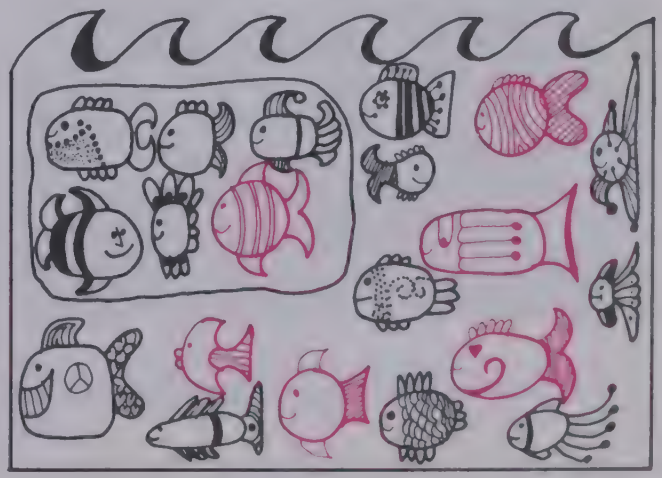
1·2 Approximations

Have you ever wondered how Fish and Game Association authorities determine the number of fish in a lake? First, they net, tag, and return to the lake some fish. After an appropriate length of time to allow the fish to become thoroughly mixed throughout the lake they net some more fish. Now, the ratio of the number of tagged fish retrieved the second time to the total number of fish caught the second time should be the same as the ratio of the fish tagged to the total number of fish in the lake.

Tagged fish



Second netting



$$\frac{(T_S)\text{ number of tagged fish caught second time}}{(F)\text{ number of fish caught second time}} = \frac{(T)\text{ number of fish tagged}}{(x)\text{ total number of fish in lake}}$$

$$\frac{T_S}{F} = \frac{T}{x}$$

As an example, let us assume

$$T = 615$$
$$T_S = 12$$
$$F = 792$$
$$x = \blacksquare$$

We make the appropriate replacements in the formula:

$$\frac{T_S}{F} = \frac{T}{x}$$
$$\frac{12}{792} = \frac{615}{x}$$

Solve for x: $$x = \frac{792 \times 615}{12}$$
$$x = \blacksquare$$

Let us approximate this answer by rounding off the numbers to one significant figure each:

$$792 \longrightarrow 800$$
$$615 \longrightarrow 600$$
$$12 \longrightarrow 10$$
$$\therefore x \simeq \frac{800 \times 600}{10}$$
$$x \simeq \blacksquare$$

This approximation is useful in deciding whether or not a given answer is reasonable. We will use approximations much more when we use the slide rule.

EXERCISE 1·2

1. Use the symbols T_S, F, and T to write a formula for x.

2. Fish and game officials made this record when testing a lake:

   July 2: 786 fish tagged.
   August 2: 468 fish netted.
   August 2: 19 tagged fish netted.

   (a) Use rounded off numbers to approximate the number of fish in the lake.

   (b) Use the exact numbers to determine the number of fish in the lake.

3. In a survey, Mark asked $\frac{1}{12}$ of the people in his school what they thought of the current prime minister. 28 people replied they think he is "a better prime minister than the previous one".

   (a) How many people in the school think the prime minister is better than the previous one, if this is a fair sample of all the people?

   (b) Check your answer by approximations.

4. An aquarium is 4'6" × 3'2" × 2'7". If a gallon of water occupies 277.77 cubic inches, how many gallons of water does the aquarium hold? (Approximate your answer first.)

5. A pile of 750 sheets of paper is 52.3 mm high. What is the thickness of one sheet?

6. The populations of the five major cities in 1971 are shown.

Montreal	2,720,413
Toronto	2,609,633
Vancouver	1,071,081
Winnipeg	576,176
Ottawa	534,685

   (a) Find the approximate total population in the five cities.
   Compare your answer with those of several others near you. Are your answers all the same? Explain.

   (b) Round off the population for each city to the nearest ten thousand. Use these numbers to find the approximate total population. Compare your answer with that in (a). Is there a difference? Explain.

   (c) Repeat (b) but round to the nearest
   (i) hundred thousand
   (ii) million

7. This article appeared in a newspaper that contained two full pages on the returnable or no-deposit bottle controversy. Show the information by a circle graph for each survey area.

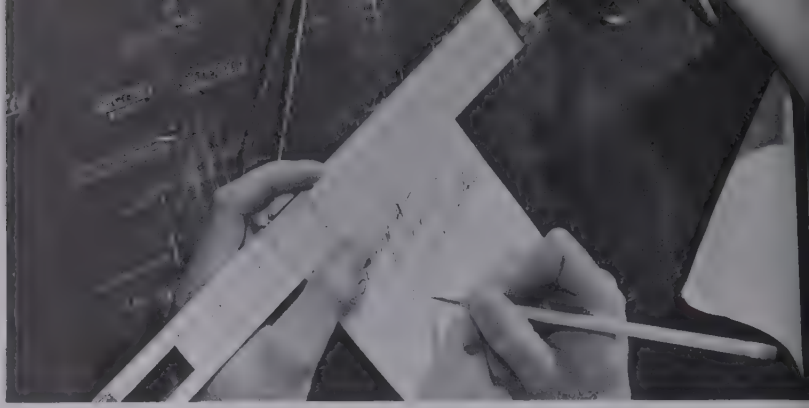
Paper big culprit

Following are the results of independent litter surveys, which indicate the proportionate contribution of various solid wastes.

Survey	Paper	Returnable Bottles	No-deposit Bottles	Cans
Dept. of Highways, Alberta	81.5%	3.54%	.86%	2.4%
Nova Scotia	92.07	.81	.27	4.2
Ontario	78.5	1.1	1.1	5.1
Metro Parks Commission, Toronto	76.3	.75	.75	7.2

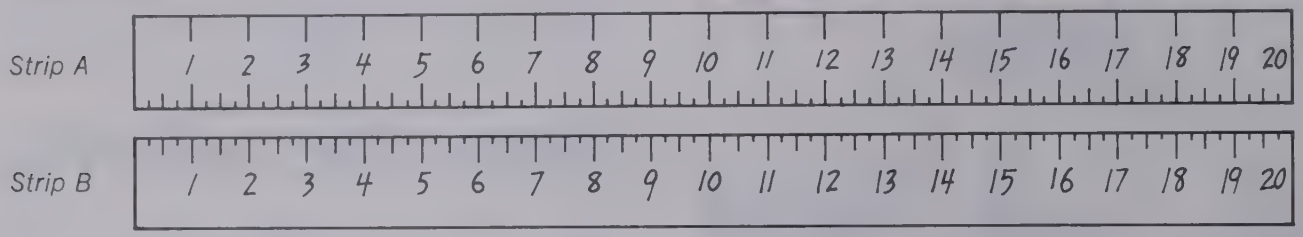
Do you think there should be emphasis placed on pollution caused by things other than bottles and cans?

1·3 Slide Rule

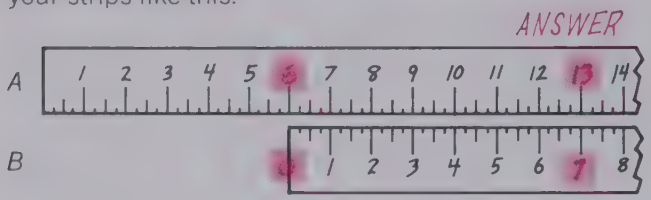


Would you like a machine to help you multiply and divide? One simple machine is the slide rule. You will have to think in order to use the slide rule.

First let us make a slide rule for adding and subtracting. Make two strips of cardboard labelled as shown. One-centimetre units are suitable. Mark the half and quarter units.

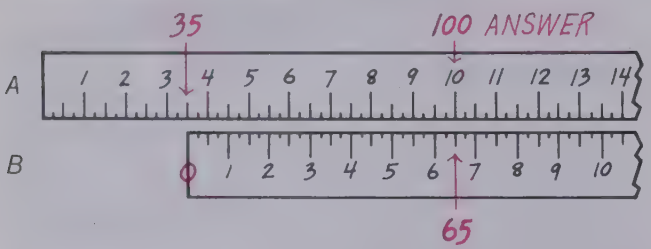


Let's use the addition slide rule to add 6 and 7. Place your strips like this:



The answer is 13.

Use your slide rule to add 35 and 65.

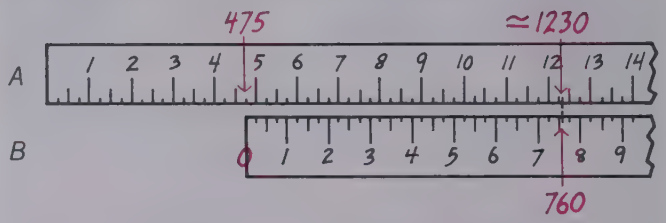


The answer is 100.

3456789

012345678

Use your slide rule to add 475 and 760.



The answer is a little more than 1225—perhaps 1230 or so.

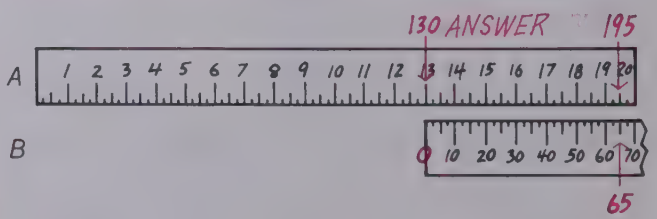
Hence we make a quick look at the question:

475	Since the sum ends in 5 and it
760	is slightly more than 1225, it
5	is 1235.

Use the slide rule to subtract: 195 – 65.

Hint!

Remember if 195 – 65 = x, then x + 65 = 195.

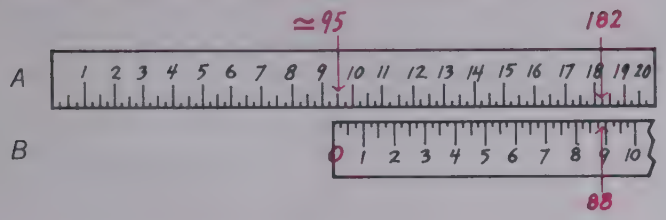


The answer is 130.

Use the slide rule to subtract: 182 – 88.

Hint!

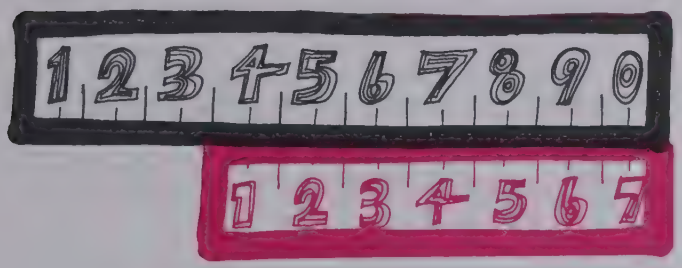
182 – 88 = x can be written x + 88 = 182.



The answer is a little less than 95.

Since 182
88
4 The answer is 94.

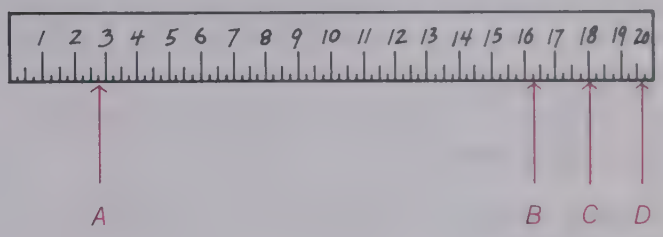
Now it is not likely you will use this slide rule for adding and subtracting. But you will need practice working with the scale. Do the exercise that follows.



EXERCISE 1·3

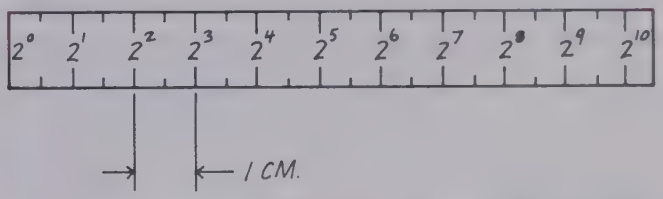
Use your slide rule to find the answers.

1. 7 + 9
2. 15 + 3
3. 45 + 125
4. 75 + 65
5. 55 + 80
6. 325 + 475
7. 775 + 650
8. 13.5 + 4.5
9. 82.5 + 45.0
10. 78.5 + 64.5
11. 42.5 + 65.6
12. 0.136 + 0.44
13. 18 – 6
14. 18.5 – 8.25
15. 19.25 – 7.75
16. 875 – 325
17. 18.4 – 6.35
18. 164 – 132
19. 1880 – 765
20. 148 – 49
21. Name three different numbers each point might represent on the slide rule.



1·4 Slide Rule for × and ÷

Make two strips of cardboard labelled as shown:



We can use this to multiply and divide certain numbers.

Example 1

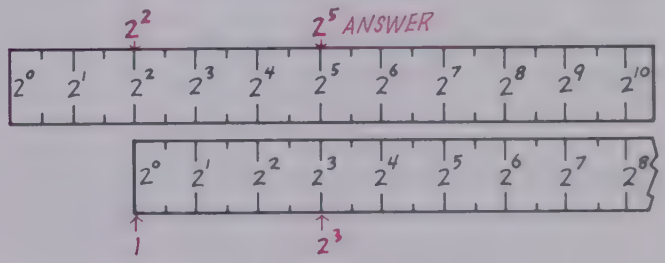
Simplify:

(a) $2^2 \times 2^3$ (b) $2^8 \div 2^3$

Solution

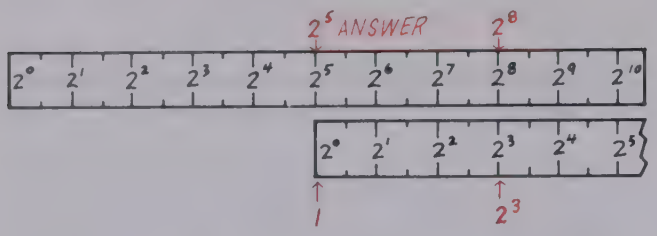
(a) $2^2 \times 2^3$.

Arrange the strips as shown:



The answer is 2^5.

(b) $2^8 \div 2^3$. Arrange the strips as shown.



Recall: $2^8 \div 2^3 = x$ implies $x \times 2^3 = 2^8$
The answer is 2^5.

Before proceeding further let us review briefly the use of exponents.

Recall: $10^2 = 10 \times 10$;
$10^5 = 10 \times 10 \times 10 \times 10 \times 10$

Hence
$$10^5 \times 10^2 = (10 \times 10 \times 10 \times 10 \times 10) \times (10 \times 10)$$
$$= 10^7$$

We conclude

$$a^m \times a^n = a^{m+n}$$

Also
$$\frac{10^5}{10^2} = \frac{10 \times 10 \times 10 \times 10 \times 10}{10 \times 10}$$
$$= 10^3$$

We conclude

$$\frac{a^m}{a^n} = a^{m-n}$$

Observe this example:
$$\frac{10^3}{10^4} = 10^{3-4}$$
$$= 10^{-1}$$

and
$$\frac{10^3}{10^4} = \frac{10 \times 10 \times 10}{10 \times 10 \times 10 \times 10}$$
$$= \frac{1}{10}$$

Hence
$$10^{-1} = \frac{1}{10}$$
$$10^{-1} = 0.1$$

$$a^{-1} = \frac{1}{a}$$

Three other rules are reviewed:
$5^2 \times 4^2 = (5 \times 4)^2$ (Confirm this)

We conclude

$$a^m \times b^m = (ab)^m$$

From
$$\frac{5^2}{4^2} = \left(\frac{5}{4}\right)^2$$

We conclude

$$\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$$

From
$$(10^2)^3 = 10^2 \times 10^2 \times 10^2$$
$$= 10^6$$

We conclude

$$(a^m)^n = a^{mn}$$

Note:
$$\frac{3^2}{3^2} = 3^{2-2}$$
$$= 3^0$$

and
$$\frac{3^2}{3^2} = \frac{3 \times 3}{3 \times 3}$$
$$= 1$$

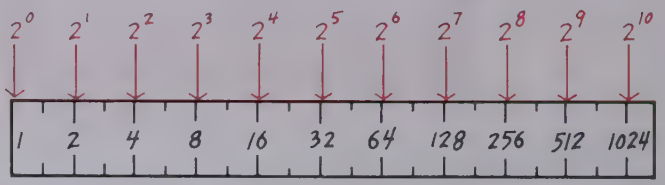
Hence $3^0 = 1$

and

$$a^0 = 1$$

Before proceeding with the next slide rule, do questions 1 and 2 in Exercise 1·4.

To make a more useful slide rule, prepare *two* strips of cardboard with these numbers. Notice the relationship of these numbers to the numbers in color.

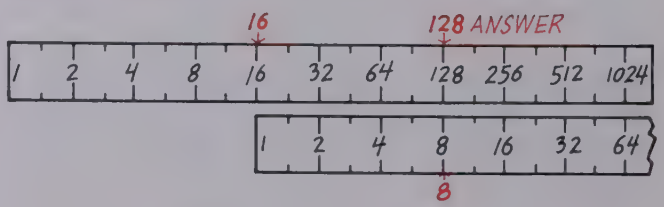


Example 2

Use this new slide rule to calculate:
(a) 16×8 (b) $512 \div 32$.

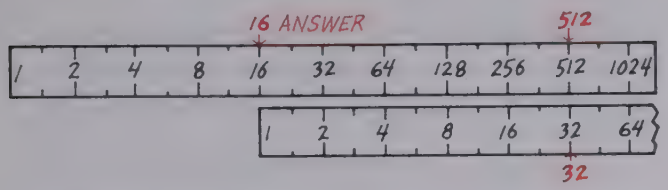
Solution

(a) Set the slide rule as shown.



The answer is 128.

(b) $512 \div 32 = x$ may be written $x \times 32 = 512$.



The answer is 16.

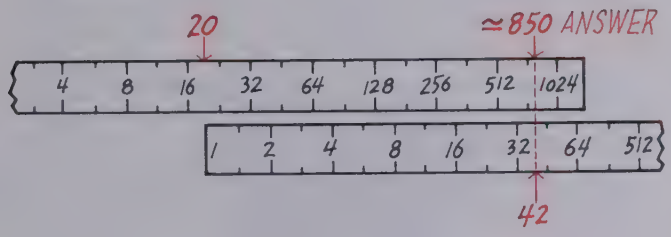
Before proceeding do questions 3 through 10 in Exercise 1·4.

Example 3

Use your slide rule to approximate each:
(a) 20×42 (b) $426 \div 95$.

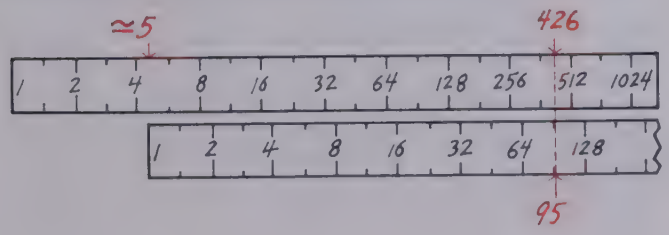
Solution

(a) 20×42 Approximate: $20 \times 40 = 800$
Set your slide as shown. Approximate the positions of 20 and 42. Then approximate the answer.



Approximating 20 we note 20 is a bit more than 16. Also we note 42 is a bit more than 32. Noticing the position of the answer (indicated by the color dotted line) we observe the answer lies between 512 and 1024. It is slightly over the half-way mark—and combining with this our approximation of 800, we approximate our answer at 850.

(b) $426 \div 95 = x$ may be written $x \times 95 = 426$.



Notice how we approximated the positions of 426 and 95. Then we approximate the answer 5. We can make a quick mental calculation to approximate the answer to check this:

$$\frac{400}{100} = 4$$

Hence our answer is in the neighborhood. Using long division, find $426 \div 95$.

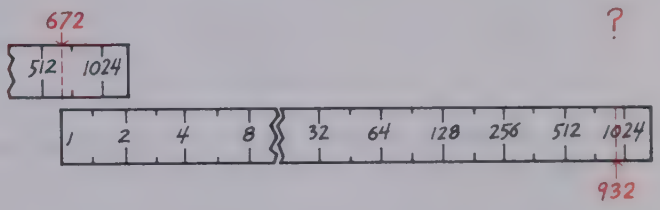
Before proceeding do questions 11 through 18 in Exercise 1·4.

Example 4

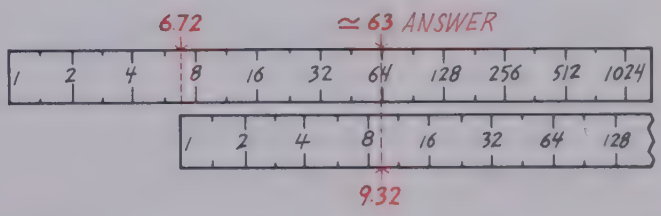
Calculate: 672×932

Solution

Approximate: $700 \times 900 = 630{,}000$.
Can you arrange your slide rule to do this problem? Obviously when we match these numbers on the slide rule as we have done previously, the answer lies off the end of strip A.



Let us write 672 as 6.72×10^2 and 932 as 9.32×10^2.
We now have $672 \times 932 = 6.72 \times 10^2 \times 9.32 \times 10^2$
$= 6.72 \times 9.32 \times 10^4$
Let us now multiply 6.72×9.32 using our slide rule.



Notice how we have used the scale. Our answer is approximately 63.
Hence: $672 \times 932 \simeq 63 \times 10^4 = 630{,}000$.

Complete Exercise 1·4.

EXERCISE 1·4

1. Use the laws of exponents to simplify:

   (a) $4^2 \times 4^3$ (b) $5^3 \times 5^4$

   (c) $15^3 \times 15^6$ (d) $\frac{10^6}{10^2}$

   (e) $4^2 \times 6^2$ (f) $\frac{15^2}{15^2}$

   (g) $\frac{a^{12}}{a^8}$ (h) $\frac{5^6}{10^6}$

   (i) $\frac{12^5}{6^5}$ (j) $\frac{12^3}{12^3}$

   (k) $\frac{14^5}{14^5}$ (l) 9^0

   (m) $\frac{4^5}{4^7}$ (n) $\frac{10^4}{10^6}$

   (o) $\frac{16^2}{16^2}$ (p) $(4^3)^2$

   (q) $(8^4)^2$ (r) $\left(\frac{1}{5}\right)^2 \times 5^2$

   (s) $\frac{5^3}{5^7}$ (t) $\frac{4^0}{4^2}$.

2. Use your first × and ÷ slide rule to calculate the following:

   (a) $2^4 \times 2^5$ (b) $2^5 \times 2^3$

   (c) $2^8 \div 2^2$ (d) $2^6 \div 2^4$

   (e) $2^{10} \div 2^5$ (f) $2^7 \div 2^7$

   (g) $2^9 \div 2^4$ (h) $2^8 \div 2^3$.

Use your second × and ÷ slide rule to calculate:

3. 4×32 4. 16×32

5. 32×32 6. 128×8

7. $512 \div 16$ 8. $512 \div 64$

9. $1024 \div 64$ 10. $256 \div 32$

11. 25×25 12. 45×14

13. 12×55 14. 35×25

15. $750 \div 430$ 16. $625 \div 90$

17. $339 \div 27$ 18. $514 \div 54$

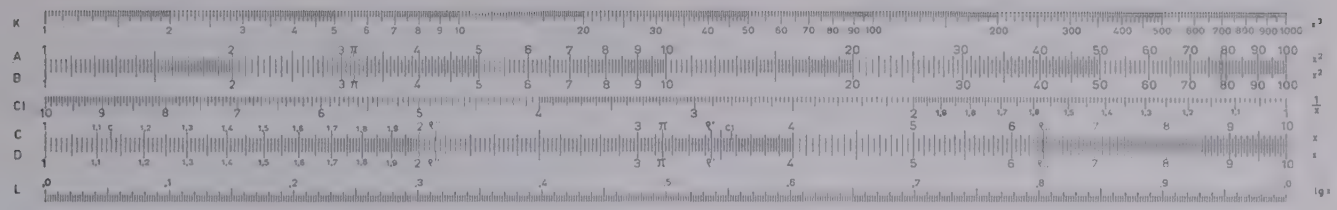
19. 64×64 20. 128×128

21. 32×128 22. $725 \div 326$

23. 625×47 24. $286 \div 13$.

25. In 1971 Calgary (population 381,000) reported 72 cars abandoned per week. How many cars per day, per year were abandoned? On this same scale, how many cars would be abandoned in Toronto (population 2,600,000) per day, per week, per year? What do you think should be done with abandoned cars? Account for the cost of implementing your plan.

It started out in a 1962 showroom as a sight for sore eyes, but by 1971 it was just an eyesore, abandoned on a city street. Then it was towed away, not to be dumped somewhere else, but to be compacted into the shape you see here. Just look around. Look at the recycling we'll need. Cans, bottles, plastics, paper, garbage, all that stuff you put out on the curb once or twice a week.



1·5 Standard Notation and Its Use

You possibly realized in the last example and the last assignment in the previous section that there is a need to know about standard notation (often called scientific notation).

Start here

$$2400 = 2.4 \times 10^3$$
$$240 = 2.4 \times 10^2$$
$$24 = 2.4 \times 10^1$$
$$2.4 = 2.4 \times 10^0$$
$$.24 = 2.4 \times 10^{-1}$$
$$.024 = 2.4 \times 10^{-2}$$
$$.0024 = 2.4 \times 10^{-3}$$

It is noted that 2×10^2 is different from 2.00×10^2. Earlier we discussed "rounding off". When we write 2.00×10^2 we mean 200 rounded off to the nearest *one*. Hence we know

$$199.5 \leq 200 < 200.5$$

When we write 2×10^2 we mean 200 rounded off to the nearest *hundred*. Hence we know

$$150 \leq 200 < 250$$

Example 1

Write each numeral in standard notation:

(a) 176 (b) 25600
(c) 0.143 (d) 0.045

Solution

(a) $176 = 1.76 \times 10^2$
(b) $25600 = 2.56 \times 10^4$ (We assume this number, 25600, is correct to the nearest hundred.)
(c) $0.143 = 1.43 \times 10^{-1}$
(d) $0.045 = 4.5 \times 10^{-2}$

Now do questions 1 and 2 in Exercise 1·5.

Let us now return to the slide rule. It is interesting to note that Henry Biggs, in 1614 invented a scale similar to that we used in Section 1-4. Edmund Guntor, in 1620, used it in a manner similar to how we used it. The first slide rule was invented by Reverend Oughtred in 1632. Sir Isaac Newton added a runner to Oughtred's rule in 1675. Yet it was not until 1850 that a French artillery officer, Amédée Mannheim, reinvented what is the modern version of the slide rule. In recognition of the French officer the instrument is called the Mannheim slide rule.

We can use scales C and D or A and B to multiply and divide. Since C and D allow greater accuracy we shall use these two. Let us look at scales C and D. They are identical.

Notice that the number divisions between various numbers differ. They are the same for 1 to 2, for 2 to 4, and 4 to 10. Before proceeding do question 3 in Exercise 1·5.

Let us use the slide rule now to multiply.

Example 2

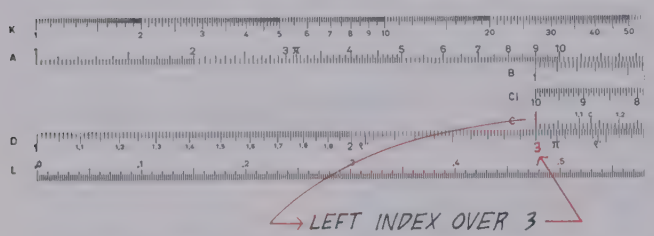
Multiply: (a) 3×2 (b) 76×82.

Solution

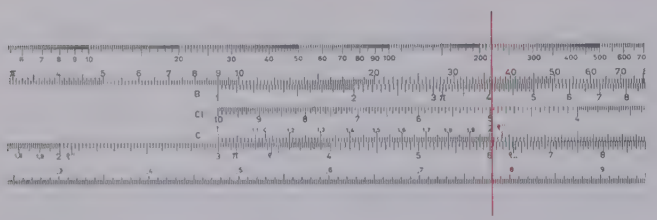
(a) 3×2

1. Estimate the answer by using standard notation if necessary. 3×2 is less than 10.

2. Set either left or right index of the C scale above one of the factors of the question.



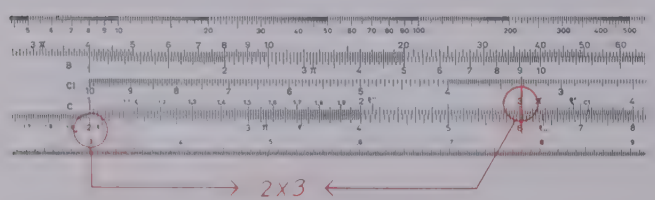
3. Move the runner until the hairline is over the other factor on the C scale.



4. Under the hairline on the D scale read the digits in the product (usually 3 digits at the most). In this case the digit is 6.

5. Place your decimal point according to your earlier approximation – step 1. In this case the decimal is after the 6.

6. It is advisable when learning to use the slide rule to check by reversing the order in which the factors were used.



(b) 76×82

Follow the six steps in part (a).

1. To approximate, write the problem in standard notation.

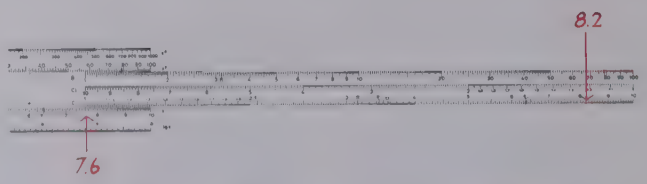
$$76 \times 82 = 7.6 \times 10^1 \times 8.2 \times 10^1$$
$$= 7.6 \times 8.2 \times 10^2$$
$$\simeq 8 \times 8 \times 10^2$$
$$\simeq 6.4 \times 10^3$$

2 and 3. We will use the statement

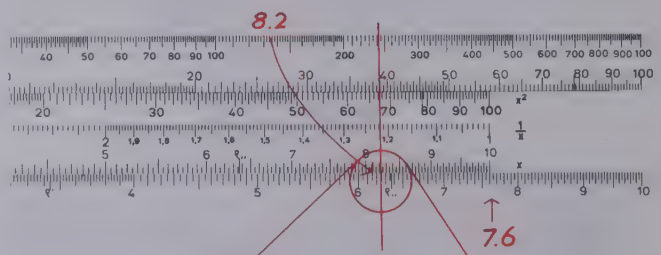
$$7.6 \times 8.2 \times 10^2$$

on the slide rule.

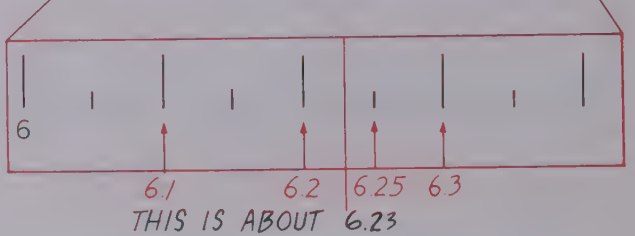
Set the left index of the C scale over the factor 7.6 on the D scale. Place the hairline at 8.2 on the C scale. Can you do this? NO! 8.2 on the C scale is far off the right end of the D scale.



In this case we must place the *right* index of the C scale over the factor 7.6 on the D scale. Now place the hairline at 8.2 on the C scale.



4. Read the digits on the D scale under the hairline. We are able to read 62 __.
The third digit is not immediately read visually. We must approximate this digit.



5. Hence the answer is 6.23×10^3.
   Compare this with the approximation in step 1. We can write this also as 6230.

$$76 \times 82 \simeq 6230$$

   (You may wish to multiply 76×82 longhand to see how accurate you are.)

6. This step will be left for you to do.

Do question 4 in Exercise 1·5 before proceeding.

Division is as easily done as multiplication.

Example 3

Divide: $455 \div 83.6$

Solution

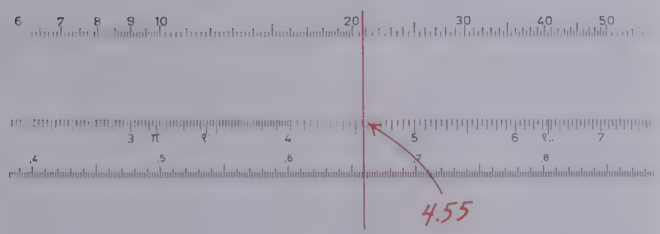
1. Approximate:

$$\frac{455}{83.6} = \frac{4.55 \times 10^2}{8.36 \times 10^1}$$

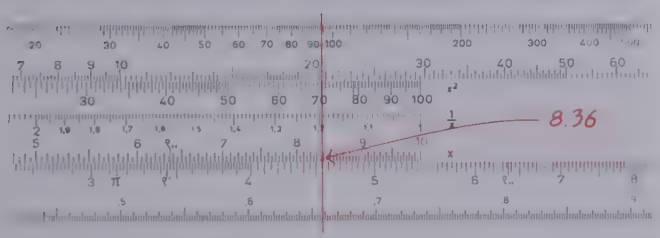
$$\simeq \frac{5}{8} \times 10$$

$$\simeq 6$$

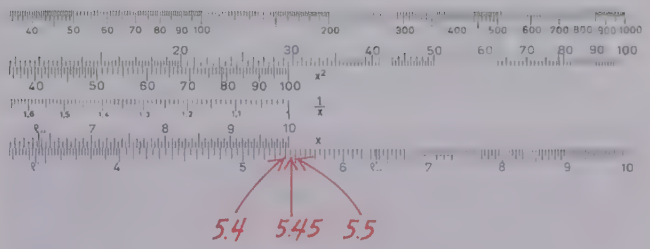
2. Use $\frac{4.55}{8.36} \times 10$
   Move runner until hairline is over 4.55 on D scale.



3. Set slide so that the divisor (8.36) is under the hairline (matching 4.55). Notice we use the visual method to locate 8.36.



4. Read the digits on the D scale under the index on the C scale.



   The digits visually are 545.

5. Place the decimal correctly according to the approximation in step. 1. Our approximation was 6 hence

$$\frac{455}{83.6} \simeq 5.45$$

6. Check by multiplying the answer (quotient) by the divisor. Your answer should be the dividend.

$$83.6 \times 5.45 = \blacksquare$$

Complete Exercise 1·5.

EXERCISE 1·5

1. Write in standard notation:

(a) 342	(b) 6724	(c) 72.3
(d) 0.14	(e) 562.7	(f) 200
(g) 200.0	(h) 4.7	(i) 0.0014
(j) 0.039	(k) 0.0009	(l) 0.056.

2. Write in decimal notation:

(a) 1.45×10^2	(b) 3.24×10^3
(c) 5.96×10^3	(d) 4.44×10^{-1}
(e) 5.76×10^{-2}	(f) 2.75×10^{-3}
(g) 3.984×10^2	(h) 3.750×10^2.

3. Locate these numbers on the slide rule by placing the hairline over each in turn. Work with a partner and let him check each answer.

(a) 1.24	(b) 2.67	(c) 3.46
(d) 4.15	(e) 4.05	(f) 6.35
(g) 8.75	(h) 9.8	(i) 1.39

4. Copy and fill in this table. Did you notice how simple it is to use a slide rule to multiply one factor by several multiplicands?

x	32	12	0.3	2.9	0.52	0.66
15						
1.8						
2.34						
56						
625						

5. Use your slide rule to calculate the quotient in:

(a) $\frac{245}{63}$ (b) $\frac{245}{22}$ (c) $\frac{375}{33}$ (d) $\frac{375}{65}$

(e) $\frac{37.5}{65}$ (f) $\frac{3.75}{6.5}$ (g) $\frac{0.84}{4.3}$ (h) $\frac{78400}{3650.}$

6. A medical lab technician takes a blood count by placing a sample of blood in a solution under a microscope. She counts the number of red blood cells observed on a certain area. She multiplies this number by 10^4. This gives the number of red blood cells per cubic millimetre. A certain technician counts 455 cells. What is the number of red blood cells per cubic millimetre in the donor's blood? Write the answer in as many ways as you can.

7. A technician counted 430 cells. Write in as many ways as you can the *blood count* of the donor.

8. A technician counted 400 cells. Write in as many ways as you can the *blood count* of the donor.

9. A floor is to be tiled. It is 28.5 feet by 34.6 feet.

   (a) What is the area of the floor?

   (b) The cost of tiling is \$1.25 per square foot. What is the cost of tiling the floor?

10. A circular flower bed is 27 feet in diameter. What is the circumference of the bed? (Hint: $C = \pi D$; $\pi \simeq 3.14$).

11. A secretary is paid \$2.75 per hour. She works from 8:30 a.m. to 1:00 p.m. five days a week. What is her weekly wage?

12. Canada has 23 national parks with a total area of approximately 31,000 square miles. Each square mile contains 640 acres.
Canada's population is approximately 21,000,000.

Determine:

(a) the average number of square miles in each park,

(b) the average number of people per square mile of park,

(c) the average number of square miles of parkland per person,

(d) the number of acres of parkland,

(e) the average number of people per acre of parkland,

(f) the average number of acres of parkland per person,

(g) the number of acres in a football field. (An acre is 160 square rods. A rod is $5\frac{1}{2}$ yards.)

(h) Do you think there is enough national parkland in Canada? Defend your answer.

(i) Research to find the number of provincial parks in your province. Work out the answers to questions (a) through (f) for your provincial park situation.

1·6 Collecting, Presenting and Interpreting Data

Project

Project 1

Work in a group of 9 students.
Use a clock or watch with a second hand.
Each student in turn sits in a chair with his forearm on the desk with hand open upward. Each student records in Table 1 the number of times the student can close and open (flexes) his hand in 15 seconds.

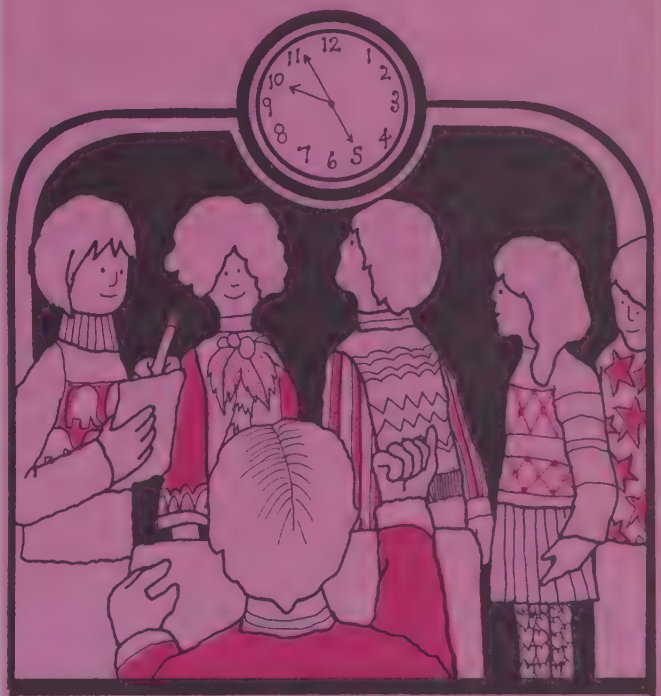


TABLE 1

STUDENT	NO. OF FLEXES
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
9.	
TOTAL	

A. Graphing

Record in as many different graphical forms as you can the information in Table 1.

B. Arithmetic Mean (Average)

Total the number of times the group flexed its hands. Divide the total by the number of students (9).

$$\text{Arithmetic Mean} = \frac{\text{Total of all scores}}{\text{No. of recordings}}$$

(i) Show the Arithmetic Mean as an extra colored bar on your bar graph in A. Label this bar.

(ii) Show the Arithmetic Mean on your other graphs also. Think about how you will do this before starting.

C. Median

Record the number of times the students flexed their hands in numerical order from lowest to highest. What number of flexes has as many above it as below it? This number is the *median*.

D. Mode

Look at the list in C. Did any number of flexes occur more often than others? This number is the *mode*. What is the mode of your data?

Example 1

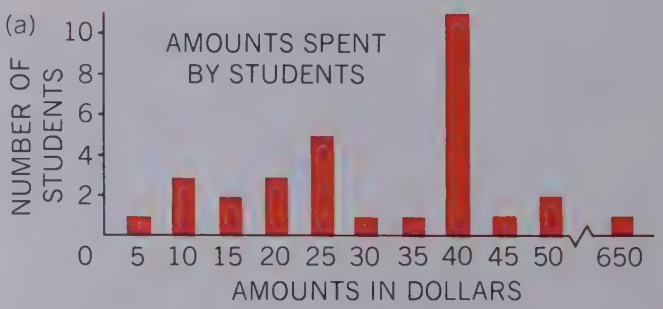
A class recorded the amount of money each student spent on lunches, car expenses, recreation, etc. during the current month. This chart shows the amounts.

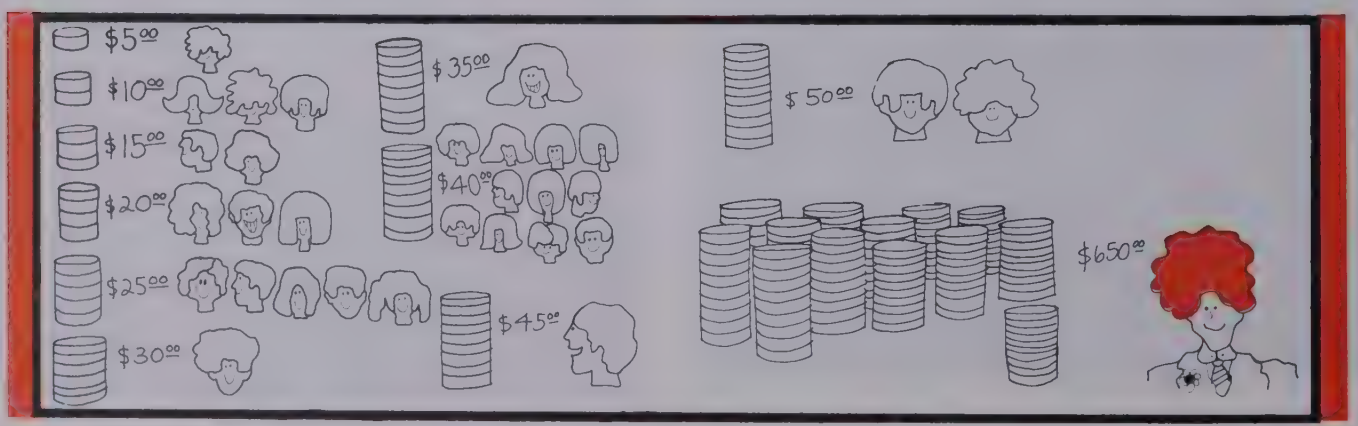
Amount to nearest $5	Tally Count	Frequency	Totals
$ 5	/	1	$ 5
$ 10	///	3	$ 30
$ 15	//	2	$ 30
$ 20	///	3	$ 60
$ 25	𝍸	5	$125
$ 30	/	1	$ 30
$ 35	/	1	$ 35
$ 40	𝍸 𝍸 /	11	$440
$ 45	/	1	$ 45
$ 50	//	2	$100
$650	/	1	$650
		31	$1,550

(a) Record the data in suitable graphical form.
(b) Find the arithmetic mean, median and mode.

Solution

(a)





Can you represent this information in another kind of bar graph?
Represent this data in other graphic forms.

(b) $$\text{Arithmetic Mean} = \frac{\text{Total}}{\text{No. of students}}$$

$$= \frac{1550}{31}$$

$$= 50$$

The arithmetic mean is \$50. This, in common terms, is called the average amount spent by the students.

Counting the number in each category we find 15 students spent more than \$35 and 15 students spent less than \$35. Therefore the median amount spent is \$35.

More students spent \$40 than any other amount. The mode is \$40. (See the bar graph.)

EXERCISE 1·6

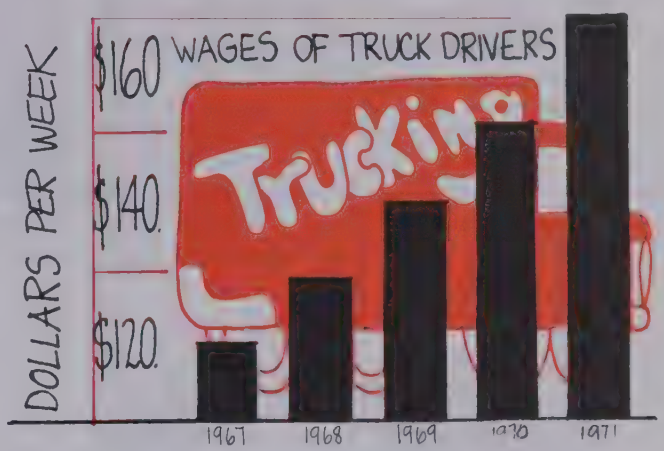
1. Refer to Example 1. Do you think the statement "The average amount spent per student was \$50" describes the situation very well? Explain.

2. The common term *average* has a number of meanings. Some people use *average* to mean *arithmetic mean*, others use it to mean *median*, while others use it to mean *mode*.

   (a) Which meaning(s) would you use if you were a member of the class (Example 1) and you were trying to convince your father to increase your allowance from the present (i) \$35 (ii) \$45 (iii) \$50?

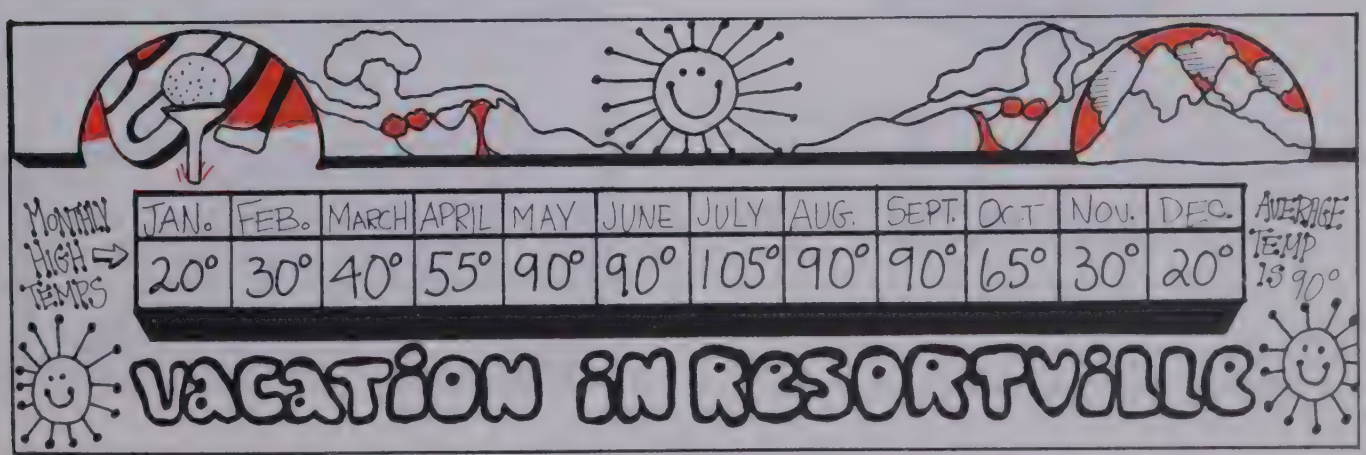
   (b) Which meaning would you use if you were a sincere and fair father of a youngster asking for an increase if the present allowance was (i) \$35 (ii) \$45 (iii) \$50?

3. Watch the newspapers and magazines for graphs or data that may lead to false impressions by the reader. Explain what is wrong with each of these:



Who might publish such information as in this graph? Why would truck drivers take exception to this information?

In the ad below, upon what basis are the data and comment correct? Do you think the ad leaves an honest impression? Why?
Who do you think would publish such an ad? Why would one publish such an ad?


MONTHLY HIGH TEMPS ⇨
JAN. FEB. MARCH APRIL MAY JUNE JULY AUG. SEPT. OCT NOV. DEC.
20° 30° 40° 55° 90° 90° 105° 90° 90° 65° 30° 20°
AVERAGE TEMP IS 90°
VACATION IN RESORTVILLE

4. Make a survey of your class to find the age in months of the class members. Make a tally count, a frequency table and suitable graph to show the data. Calculate the arithmetic average, the median and the mode.

5. Data are used in many ways. Record the marks on the last test of those students who smoke cigarettes. Calculate the average mark for this group. Then record the marks on the same test of those students who do not smoke cigarettes. Calculate the average mark for this group.

   If you are a smoker, can you make a statement based on some interpretation of this data, that smoking does not adversely affect marks in school?

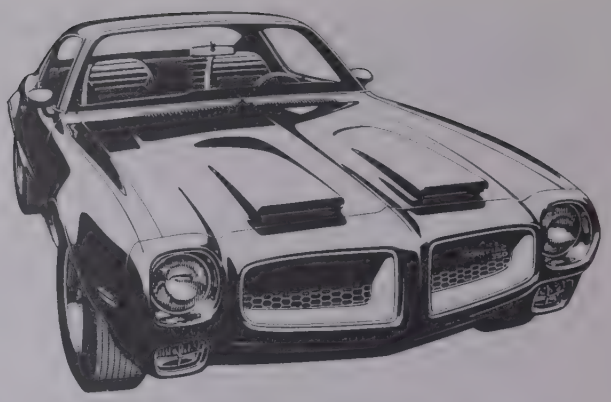
   If you are a non-smoker, can you make a statement based on some interpretation of this data, that smoking does adversely affect marks in school?

6. Repeat question 5 to make statements about marks in school of students who drive cars as compared with those of students who do not drive cars. Can you derive some "average" which would support a claim that

   (a) students should drive cars

   (b) students should not drive cars

   if they want to get high marks in school?



7. The enrolment in 15 high schools is shown:

B.C.H.S.	567	S.W.C.H.S.	432
C.H.H.S.	1326	C.M.H.S.	245
E.C.M.H.S.	451	L.B.H.S.	764
J.F.H.S.	1332	Q.E.H.S.	627
W.C.H.S.	1331	F.L.H.S.	433
W.A.H.S.	1334	E.P.S.H.S.	247
H.W.H.S.	1884	J.D.H.S.	230
V.B.H.S.	1042		

   (a) Round off the enrolment to the nearest ten students. Use the rounded off figures for the balance of this question.

   (b) Set up a tally count of the number of schools to show enrolment.

   (c) Set up a frequency table showing the data.

   (d) Draw a suitable graph to "tell" the facts honestly. Have a contest to see who can present the information in the most interesting way.

   (e) Calculate the arithmetic mean, median and mode.

   (f) Assume you are each of these persons in turn. How would you interpret the data and "average"?

      (i) A school board member wanting to increase enrolments.

      (ii) A parent whose son goes to W.A.H.S., complaining about "too large a school".

      (iii) You are a student attending Q.E.H.S. and you complain the school is too small for adequate course offerings.

   (g) Endeavour to draw a graph to support graphically the claims of each person in (f). You must use the actual data.

Using the Library

Obtain the book *How to Lie With Statistics* by Darrell Huff. Read Chapter 10, pages 122-142. (You may want to read more—such as chapter 8). What questions should you ask yourself when you are endeavouring to interpret and reason through facts supported by statistics?

While computation is essential to mathematics there is more to math than *figuring*. Math is also concerned with proper and appropriate use of its many concepts. Reasoning and logic are a vital part of mathematics. Many people, when confronted with "figures" or statistics, consider them to be facts. The book *How to Lie With Statistics* is a must for everybody's reading.

Discuss each picture. What is wrong with the logic or implied message in each picture?

Who is the Average?
What is the Average?

Labour Minister: "Our efforts since July to curb unemployment by providing jobs have brought about a decrease in the number of unemployed."

REVIEW EXERCISE 1

1. Round off each number to the nearest hundred.

   (a) 1454.6 (b) 3496 (c) 8722

2. Use your slide rule to make the following calculations:

   (a) 442×673 (b) $\frac{647}{28}$

   (c) 0.417×6.3 (d) $\frac{46.3}{84.5}$

3. The chart shows the average precipitation in inches in 11 months in Grand Banks, Newfoundland.

J	F	M	A	M	J	J	A	S	O	N
4.7	4.5	4.5	3.5	4.1	4.1	3.5	3.5	4.2	5.8	6.0

   (a) Show the data by a graph.

   (b) Find the arithmetic mean, median and mode.

   (c) Pretend you run tourist accommodation in Grand Banks. Based on the statistics make a statement about the precipitation.

Projects

Project 1

It is claimed that phosphates contribute to algae pollution in rivers and lakes. Phosphates do not break down in sewage disposal systems. Phosphates are an ingredient in detergents.

With the permission of the manager, visit a supermarket and make a record of the percent of phosphates in the various detergents on the market. Make circle graphs to show percent phosphates in each detergent.

Find the three *averages* discussed in this chapter. Use the appropriate information to write an ad for:

(a) the detergent with the least amount of phosphate

(b) the detergent with a medium amount of phosphate

(c) the detergent with the greatest amount of phosphate

(d) the active antipollution group.

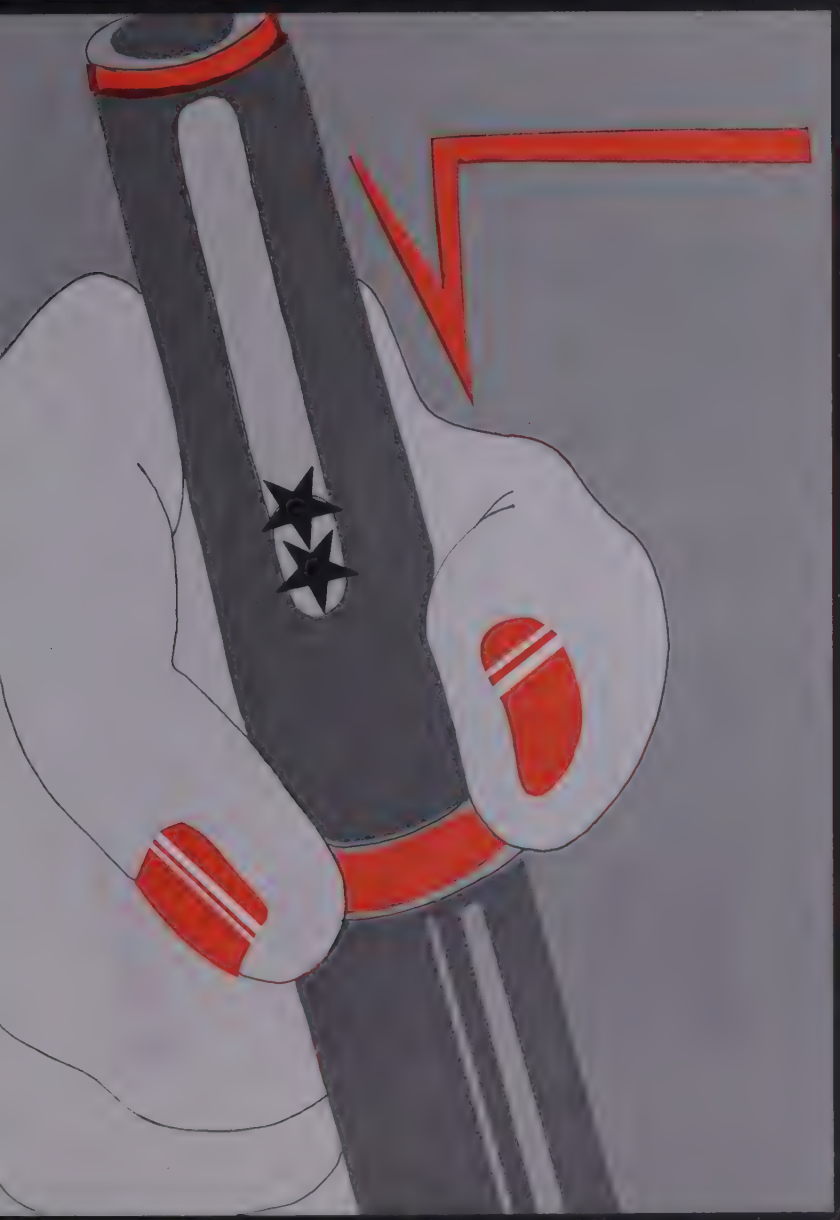
Project 2

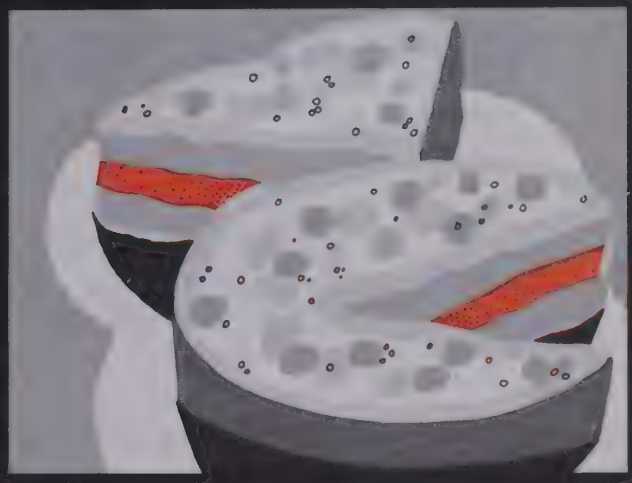
Keep a record of the number of livestock of various kinds and the prices at the local stockyards. This information is available from the radio, newspapers or directly from the stockyards. Try to show the data in a way that will help a stockyard producer to decide at what time of the week, month or year he should sell his product. This may require several graphs that *overlay* one another.

Project 3

Obtain from the police department the record of automobile accidents showing the time of the accidents over the period of a month. Try to determine when the most accidents occur, when the most serious (personal injuries) and most costly (property damage) happen. Represent the data and findings graphically. Make up an ad using your information supporting *Safe Driving Week*.

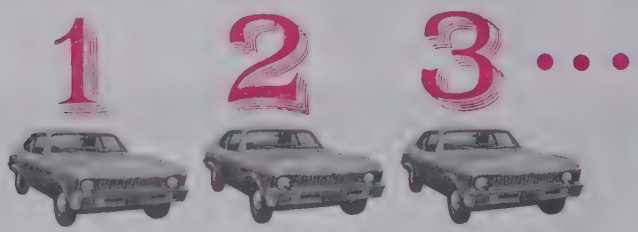
Chapter Two
IRRATIONAL NUMBERS AND SQUARE ROOTS





2·1 Counting Numbers, Whole Numbers and Integers

The set of *natural numbers* sometimes called the *counting numbers* is the set one would use if one were going to count the items in a group—cows in a herd, cars on a parking lot. A one-to-one correspondence like this would be made:



We indicate the set of natural numbers as

$$N = \{1, 2, 3, 4, \ldots\}.$$

The three dots in the above statement indicate that the set of natural numbers continues without end. To suggest the idea of no cars or no cows, the set of natural numbers is amended to include zero and becomes the set of *whole numbers*:

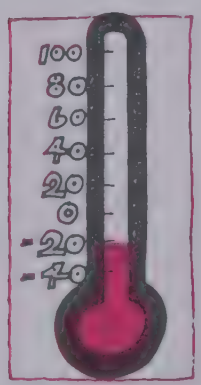
$$W = \{0, 1, 2, 3, 4, \ldots\}.$$

Algebraically, zero is necessary in order to solve problems such as these:

$45 + \blacksquare = 45$ $\qquad$ $12 - 12 = \blacksquare$

$\begin{cases} x + 3 = 3 \\ x = \blacksquare \end{cases}$ $\qquad$ $\begin{cases} 2 - x = 2 \\ x = \blacksquare \end{cases}$

The set of whole numbers is insufficient to enable one to express himself completely in mathematical language. One hears statements such as:

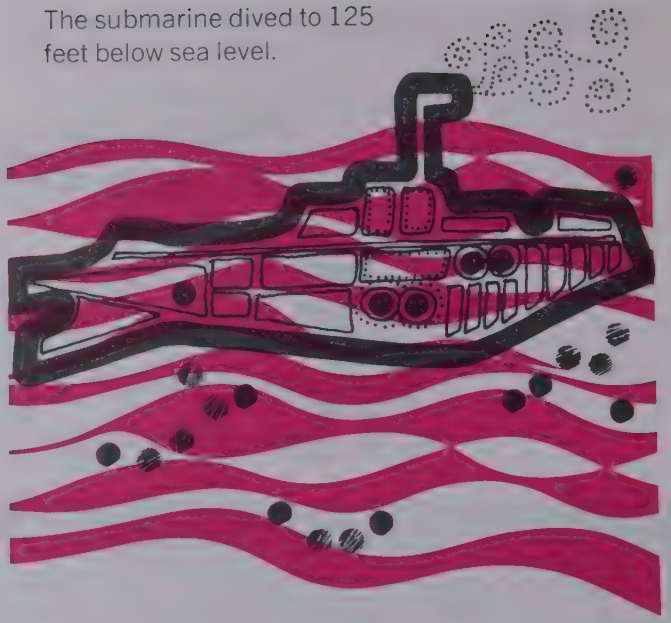


The temperature is twenty degrees below zero.

Tom withdrew fifteen dollars from his bank account.

Euclid lived about 300 B.C.

The submarine dived to 125 feet below sea level.



Consider also the solutions to problems like these:

$$30 + \blacksquare = 10$$
$$10 - 30 = \blacksquare$$

Such statements and questions illustrate the need for the set of integers

$$I = \{\ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

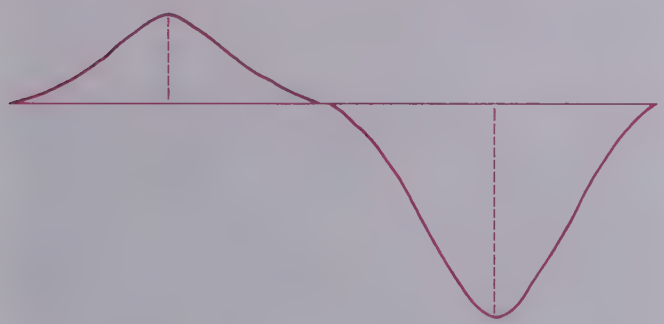
It is customary to illustrate the set of integers on the horizontal number line with the positive numbers to the right of the zero and the negative numbers to the left of the zero.

Sometimes the number line is vertical and by custom the positive numbers are placed above the zero and the negative numbers below the zero.

EXERCISE 2·1

1. Which of the following properties are true in the set of integers? Give two examples to illustrate each property that is true and one counter-example if the property is not true in the set of integers.
   (a) closure under addition
   (b) closure under multiplication
   (c) closure under subtraction
   (d) closure under division
   (e) commutative under addition
   (f) commutative under multiplication
   (g) associative under addition
   (h) associative under multiplication
   (i) associative under division

2. Mount Robson in British Columbia is 12,972 feet above sea level. The deepest trough in the Pacific Ocean is −34,210 feet *above* sea level. What is the total vertical distance between the top of Mount Robson and the deepest trough in the Pacific Ocean?



3. Simplify:
   (a) $16 - (13 - 37) + (14 - 19)$
   (b) $37 + (36 - 53 + 7) - (71 - 43 - 7)$
   (c) $44 - [42 - (7 - 84) + 6]$
   (d) $(13 - 62) - (73 - 151)$
   (e) $3(45 - 91) - [23(13 - 31) + 9(14 - 56)]$
   (f) $21(-4 + 13) + 31(23 - 41) - 3(8 - 27)$.

4. When $m = -2$, $n = 5$ find the value of:
   (a) $m(mn + m^2n - mn^2)$
   (b) $mn^2(m + n)(m - n)$
   (c) $m^2 + 2mn + n^2$
   (d) $m^3 + 3m^2n + 3mn^2 + n^3$.

2·2 Rational Numbers

In question 1 of Exercise 2·1 you found that the set of integers was not closed with respect to division. That is, in the set of integers there is no solution set for the open sentences

$$2x = 5$$
$$3x + 7 = 0 \quad \text{or} \quad 3(\blacksquare) + 7 = 0$$

To remedy this situation the set of rational numbers is introduced.

The statement of subtraction can be rewritten as one of addition:

$$13 - 7 = \blacksquare$$
$$7 + \blacksquare = 13.$$

This can be thought of as "What number added to 7 gives 13?"

In general terms:

$$a - b = c \text{ is equivalent to } b + c = a$$
$$\text{where } a, b, c \in I.$$

Subtraction is the inverse operation of addition.

The division question

$$8 \div 4 = 2$$

is equivalent to the multiplication

$$4 \times 2 = 8.$$

Division is the inverse operation of multiplication.

In general,

$$a \div b = c \text{ is equivalent to } b \times c = a.$$

But consider the case

$$15 \div 0 = \blacksquare$$

According to the general statement this should be equivalent to

$$0 \times \blacksquare = 15$$

The question here is "Zero multiplied by what number gives 15?" Obviously, the answer is "There is no such number." Then it would appear that division by 0 has no meaning. Therefore the general rule is written

$$a \div b = c \text{ is equivalent to } b \times c = a$$
$$\text{where } a, b, c \in I, b \neq 0.$$

The quotient $a \div b$ is often written $\frac{a}{b}$. Using the example we looked at earlier,

$$2x = 5,$$

we might ask "Two multiplied by what integer gives 5?" No integer satisfies this condition. Therefore the set of integers is not sufficient to satisfy our needs. We need a number that can be represented by $\frac{5}{2}$. Therefore it becomes necessary to invent the set of rational numbers.

$$Q = \left\{ \frac{a}{b} \middle| \; a, b \in I, b \neq 0 \right\}$$

EXERCISE 2·2

1. Which of the following are rational numbers?

   (a) $\frac{2}{3}$ (b) $\frac{-5}{4}$ (c) $\frac{3 \times 2}{-4 \times 5}$

   (d) $\frac{\pi}{2}$ (e) $\frac{-3}{-3}$ (f) $\frac{0}{2 \times 5}$

   (g) $\frac{2 \times 3}{0}$ (h) $\frac{-4}{0}$ (i) $\frac{-\pi}{\pi}$

   (j) 4 (k) $\sqrt{\frac{9}{16}}$ (l) $\sqrt{\frac{4}{13}}$

   (m) $\frac{6}{\sqrt{2}}$ (n) $\frac{7}{\sqrt{49}}$ (o) $\frac{\pi}{\sqrt{\pi}\sqrt{\pi}}$

2. Given $H = \{0, 2, 3, 5\}$ and $J = \{1, 4, 6\}$ list all the rational numbers given by

$$K = \left\{ \frac{a}{b} \middle| \; a \in H, b \in J \right\}.$$

3. Given $M = \{-3, -2, -1, 0\}$ and $N = \{4, 5, 6\}$ list all the rational numbers given by

$$P = \left\{ \frac{a}{b} \middle| \; a \in M, b \in N \right\}.$$

4. Given $S = \{..., -2, -1, 0, 1, 2, ...\}$ and $T = \{1\}$ list some of the rational numbers given by

$$V = \left\{ \frac{a}{b} \middle| \; a \in S, b \in T \right\}.$$

   Since $\frac{4}{1}$ is equivalent to 4, the set of numbers represented by $\frac{a}{1}$, $a \in W$ is equivalent to the set of whole numbers. List several examples from the set V that are members of the set of whole numbers. What is another name for the set V?

5. Given $F = \{0\}$ and $E = \{..., -3, -2, -1, 2, 3, ...\}$ what is the set determined by the rational numbers

$$G = \left\{ \frac{a}{b} \middle| \; a \in F, b \in E \right\}?$$

6. In $Q = \left\{ \frac{a}{b} \middle| \; a, b \in I, b \neq 0 \right\}$ why do you think $b \neq 0$ is included?

7. Why do you think Q is used as the symbol for the set of rational numbers?

Project

Project 1

Pollution is a current problem. Take a *litter count* in five different halls of your school (or in one hall on five different days). What is the average litter count per hall (or per day)?

Record the *litter count* for each type of litter. On the average which type of litter had the highest count per hall (or per day)?

What is the difference between the highest and the lowest average litter count by type?

What can you do to improve the pollution problem in your school?

2·3 Basic Rational Numerals

The following are sets of equivalent fractions:

$\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots\}$ $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots\}$

$\{\frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \ldots\}$ $\{\frac{8}{9}, \frac{16}{18}, \frac{24}{27}, \frac{32}{36}, \ldots\}$

The first numeral in each case is called the basic fraction or is said to be in lowest terms. The numerator and denominator have no factor, in common, other than 1.

> A fraction is said to be in lowest terms when the numerator and denominator have no factors, in common, other than 1.

The *Multiplicative Identity Property of Rational Numbers* states

$$x \times 1 = 1 \times x = x,\ x \in Q.$$

We may multiply any rational number by 1 and still have the same number. The numeral may be different but the new numeral names the same number as the original numeral.

$$\frac{1}{2} = \frac{1}{2} \times 1 = \frac{1}{2} \times \frac{2}{2} = \frac{1 \times 2}{2 \times 2} = \frac{2}{4}$$

$$\frac{3}{5} = \frac{3}{5} \times 1 = \frac{3}{5} \times \frac{(-4)}{(-4)} = \frac{3 \times (-4)}{5 \times (-4)} = \frac{-12}{-20}$$

$$\frac{\frac{1}{3}}{\frac{2}{5}} = \frac{\frac{1}{3}}{\frac{2}{5}} \cdot 1 = \frac{\frac{1}{3}}{\frac{2}{5}} \times \frac{\frac{2}{5}}{\frac{2}{5}} = \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{2}{5} \times \frac{2}{5}} = \frac{\frac{2}{15}}{\frac{4}{25}}$$

We may reverse this operation to get a basic fraction.

$$\frac{12}{21} = \frac{4 \times 3}{7 \times 3} = \frac{4}{7} \times \frac{3}{3} = \frac{4}{7} \times 1 = \frac{4}{7}$$

$$\frac{15}{39} = \frac{5 \times 3}{13 \times 3} = \frac{5}{3} \times \frac{3}{3} = \frac{5}{3} \times 1 = \frac{5}{13}$$

In the event that the factors common to both numerator and denominator are not apparent we need only express each term as a product of its prime factors and then arrange the common factors as we did above.

$$\frac{36}{84} = \frac{2 \times 3 \times 2 \times 3}{2 \times 2 \times 7 \times 3} = \frac{3 \times 3 \times 2 \times 2}{7 \times 3 \times 2 \times 2}$$
$$= \frac{3}{7}\left(\frac{3}{3}\right)\left(\frac{2}{2}\right)\left(\frac{2}{2}\right) = \frac{3}{7} \times 1 \times 1 \times 1 = \frac{3}{7}$$

$$\frac{60}{252} = \frac{5 \times 2 \times 2 \times 3}{3 \times 3 \times 2 \times 2 \times 7} = \frac{5 \times 3 \times 2 \times 2}{7 \times 3 \times 3 \times 2 \times 2}$$
$$= \frac{5}{7 \times 3}\left(\frac{3}{3}\right)\left(\frac{2}{2}\right)\left(\frac{2}{2}\right) = \frac{5}{7 \times 3} \times 1 \times 1 \times 1 = \frac{5}{21}$$

We generalize in this statement to cover all cases:

$$\frac{xr}{yr} = \frac{x}{y}\left(\frac{r}{r}\right) = \frac{x}{y},\ x, y, r \in Q;\ y, r \neq 0$$

EXERCISE 2·3

Find a fraction in lowest terms that names an equivalent number to each of the following. Show each step.

1. $\frac{14}{21}$ 2. $\frac{36}{48}$ 3. $\frac{26}{65}$ 4. $\frac{27}{42}$ 5. $\frac{36}{60}$

6. $\frac{33}{189}$ 7. $\frac{70}{98}$ 8. $\frac{72}{216}$ 9. $\frac{144}{648}$ 10. $\frac{468}{792}$

Find a fraction in lowest terms that names each number.

11. $\frac{128}{132}$ 12. $\frac{144}{168}$ 13. $\frac{792}{2904}$ 14. $\frac{500}{700}$ 15. $\frac{324}{486}$

16. Write four fractions equivalent to each:
    (a) $\frac{3}{5}$ (b) $\frac{5}{6}$ (c) $\frac{3}{8}$ (d) $\frac{7}{12}$

17. In each set of equivalent fractions there is at least one error. Find and correct all errors.
    (a) $\{\frac{2}{7}, \frac{8}{21}, \frac{10}{28}, \frac{4}{14}, \ldots\}$
    (b) $\{\frac{3}{13}, \frac{6}{26}, \frac{12}{42}, \frac{15}{65}, \frac{13}{39}, \ldots\}$
    (c) $\{\frac{4}{9}, \frac{40}{90}, \frac{36}{80}, \frac{300}{720}, \ldots\}$
    (d) $\{\frac{4}{15}, \frac{6}{60}, \frac{24}{80}, \frac{30}{120}, \frac{6}{65}, \ldots\}$

18. A square of side 6 inches is compared to one whose side is 18 inches. How do the areas of the two squares compare?

19. Some authorities claim the tin and aluminum can is a contributor to the pollution problem. It is estimated that in 1971 the average number of cans used per person per year in Canada was 150. How many cans per day per person were used? If the population in 1971 was 21,324,000 what was the total number of cans used in Canada that year?

2·4 Working With Rational Numbers: Multiplication

Study the examples in order to review the operations with rational numbers.

Example

Find the product in each case.

(a) $\frac{3}{4} \cdot \frac{-5}{7}$ (b) $\frac{a}{3b} \cdot \frac{c}{2d}$, $b, d \neq 0$

Solution

(a) $\frac{3}{4} \times \frac{-5}{7} = \frac{3 \times (-5)}{4 \times 7} = \frac{-15}{28}$

(b) $\frac{a}{3b} \times \frac{c}{2d} = \frac{ac}{6bd}$

We now give a general definition for the multiplication of any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$.

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

EXERCISE 2·4

1. Copy and complete each statement

(a) $\frac{2}{3} \times \frac{7}{9} = \frac{\blacksquare}{27}$ (b) $\frac{6}{-5} \times \frac{3}{7} = \frac{\blacksquare}{-35}$

(c) $\frac{13}{15} \times \frac{-2}{3} = \frac{-26}{\blacksquare}$ (d) $\frac{14}{17} \times \frac{-3}{5} = \frac{-42}{\blacksquare}$

(e) $\frac{3x}{5} \times \frac{-2}{7} = \blacksquare$ (f) $\frac{5x}{7a} \times \frac{6y}{9} = \blacksquare$

(g) $\frac{7}{y} \times (-3) = \blacksquare$ (h) $(-b) \times \frac{(-7)}{7} = \blacksquare$

(i) $\frac{x}{z} \times \frac{y}{m} = \blacksquare$ (j) $\frac{5}{9} \times \frac{x}{2} = \blacksquare$

2. Find the solution set for each of the following.

(a) $\frac{3}{4} \times \frac{x}{-2} = \frac{6}{8}$ (b) $\frac{x}{3} \times \frac{9}{4} = \frac{-36}{12}$

(c) $\frac{4}{x} \times \frac{7}{6} = \frac{28}{-42}$ (d) $\frac{5}{y} \times \frac{7}{19} = \frac{35}{-57}$

(e) $\frac{7}{12} \times \frac{-4}{b} = \frac{-28}{84}$ (f) $\frac{13}{11} \times \frac{6}{a} = \frac{78}{121}$

(g) $\frac{(-3)}{4} \times \frac{a}{8} = \frac{27}{32}$ (h) $\frac{4}{7} \times \frac{x}{-3} = \frac{12}{21}$

3. Copy and complete the following statements.

(a) $\frac{2}{3} = \frac{\blacksquare}{15}$ (b) $\frac{3}{7} = \frac{\blacksquare}{21}$

(c) $\frac{4}{9} = \frac{\blacksquare}{72}$ (d) $\frac{-3}{8} = \frac{\blacksquare}{24}$

(e) $\frac{-24}{6} = \frac{\blacksquare}{1}$ (f) $\frac{14}{-7} = \frac{\blacksquare}{1}$

(g) $\frac{27}{3b} = \frac{\blacksquare}{b}$ (h) $\frac{12b}{3b} = \frac{\blacksquare}{1}$

(i) $\frac{22}{-11} = \frac{-2}{\blacksquare}$ (j) $\frac{-3}{9} = \frac{-1}{\blacksquare}$

(k) $\frac{a}{-b} = \frac{\blacksquare}{bc}$ (l) $\frac{x}{z} = \frac{xy}{\blacksquare}$

(m) $\frac{ac}{bc} = \frac{a}{\blacksquare}$ (n) $\frac{-km}{m} = \frac{k}{\blacksquare}$

(o) $\frac{-34}{2} = \frac{17}{\blacksquare}$ (p) $\frac{-k}{rk} = \frac{-1}{\blacksquare}$

4. Multiply and express in lowest terms.

(a) $(\frac{2}{3} \times \frac{-4}{3}) \times \frac{1}{7}$ (b) $(\frac{1}{13} \times \frac{-21}{2}) \times \frac{1}{3}$

(c) $(\frac{3}{-4}) \times (\frac{-2}{3} \times \frac{1}{3})$ (d) $\left(\frac{-3}{a}\right) \times \left(\frac{a}{b} \times \frac{-b}{6}\right)$

(e) $(\frac{4}{7} \times \frac{-3}{5}) \times \frac{5}{2}$ (f) $(\frac{4}{7}) \times (\frac{-3}{5} \times \frac{5}{2})$

5. A major cause of air pollution is the automobile. Make a count of cars travelling on a major route in your area during rush hours. Repeat for different days.

Count the number of cars carrying 1 person, 2 persons, 3 persons, 4 persons, 5 persons.

Collect enough information to provide a good sample.

Is there a difference between morning *rush hour* and evening *rush hour*?

(a) What percent of the cars carried 1 person? 2 persons? 3 persons? 4 persons? 5 persons?
(b) How many cars would have been needed to carry the people counted if every car had carried 4 persons?
(c) What fraction of the total cars you counted is the number needed in (b)?
(d) What conclusion can you draw from this survey?
(e) What recommendations would you make after this survey to
  (i) your family
  (ii) your community
  (iii) your school?

2·5 Working With Rational Numbers: Division

Study these examples.

Example

(a) $\frac{2}{3} \div \frac{7}{5}$ (b) $\frac{a}{3b} \div \frac{5}{2y}$, $b \neq , y \neq 0$

Solution

(a) $\frac{2}{3} \div \frac{7}{5} = \frac{2}{3} \times \frac{5}{7} = \frac{2 \times 5}{3 \times 7} = \frac{10}{21}$

(b) $\frac{a}{3b} \div \frac{5}{2y} = \frac{a}{3b} \times \frac{2y}{5} = \frac{2ay}{15b}$

General statement: Where $\frac{a}{b}$ is any rational number and $\frac{c}{d}$ is any nonzero rational number,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c} = \frac{ad}{bc}.$$

EXERCISE 2·5

1. Copy and complete the following sentences

(a) $\frac{2}{3} \div \frac{4}{7} = \frac{2}{3} \times \blacksquare$ (b) $\frac{13}{4} \div \frac{-3}{2} = \frac{13}{4} \times \blacksquare$

(c) $\frac{-4}{-5} \div \frac{3}{2} = \frac{-4}{-5} \times \blacksquare$ (d) $\frac{13}{3} \div \frac{2}{-3} = \frac{13}{3} \times \blacksquare$

(e) $\frac{3}{4} \div 2 = \frac{3}{4} \times \blacksquare$ (f) $\frac{n}{4} \div n = \frac{n}{4} \times \blacksquare$

(g) $\frac{a}{b} \div c = \frac{a}{b} \times \blacksquare$ (h) $9 \div \frac{a}{b} = 9 \times \blacksquare$

2. Find the solution set for x in each of the following:

(a) $\frac{x}{9} \div \frac{5}{2} = \frac{8}{45}$ (b) $\frac{3}{x} \div \frac{5}{8} = \frac{24}{35}$

(c) $\frac{-4}{x} \div \frac{1}{3} = \frac{-12}{5}$ (d) $\frac{3}{4} \div \frac{x}{6} = \frac{18}{20}$

(e) $\frac{3a}{-1} \div \frac{x}{5} = \frac{15a}{8}$ (f) $\frac{4b}{5c} \div \frac{3}{x} = \frac{-28b}{15c}$

3. Find the quotient in each case and specify which variables cannot equal zero. In each case write the answer in lowest terms.

(a) $\frac{3}{4} \div \frac{-7}{8}$ (b) $\frac{5}{8} \div \frac{-3}{4}$

(c) $\frac{7a}{8b} \div \frac{-4}{3b}$ (d) $\frac{13b}{49} \div \frac{396}{7}$

(e) $\frac{4x}{7y} \div \frac{-6y}{3}$ (f) $\frac{n}{x} \div \frac{4}{7}$

Project

Project 2

Shortage of water is plaguing many cities in the United States. It is predicted that Canadian cities will suffer in the near future. Do you waste water?

Are you in the habit of letting the water run while you brush your teeth? Next time you brush your teeth place a container under the tap and catch the water. Calculate the number of pints per week lost, the number of quarts per month and the number of gallons per year.

If each person in your city (community) wastes this much water, how much is lost each year?

Let us assume that one cup of water is all that is needed to brush your teeth. Compare the amount wasted to the amount needed to brush your teeth.

2·6 Working With Rational Numbers: Addition

Example 1

Find the sum of each case.

(a) $\frac{3}{5}+\frac{1}{5}$ (b) $\frac{1}{5}+\frac{-3}{10}$ (c) $\frac{11}{3}+\frac{-3}{5}$

Solution

(a) $\frac{3}{5}+\frac{1}{5}=\frac{3+1}{5}=\frac{4}{5}$

(b) $\frac{1}{5}+\frac{-3}{10}=\frac{1\times 2}{5\times 2}+\frac{-3}{10}=\frac{2}{10}+\frac{-3}{10}$

$=\frac{2+(-3)}{10}=\frac{-1}{10}$

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{b}$

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{nb}$

$$\frac{a}{b}+\frac{c}{nb}=\frac{an+c}{bn}$$

(c) $\frac{11}{3}+\frac{-3}{5}=\frac{11}{3}\times\frac{5}{5}+\frac{-3}{5}\times\frac{3}{3}$

$=\frac{55}{15}+\frac{-9}{15}$

$=\frac{55+(-9)}{15}$

$=\frac{46}{15}$

For any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$$

EXERCISE 2·6

1. Find the value of x in each case so that the statement is correct.

(a) $\frac{4}{5} = \frac{x}{20}$ (b) $\frac{3}{9} = \frac{x}{27}$

(c) $\frac{1}{4} = \frac{x}{12}$ (d) $\frac{2}{-3} = \frac{x}{24}$

(e) $\frac{4a}{5} = \frac{x}{30}$ (f) $\frac{7}{8a} = \frac{x}{16a}$

(g) $\frac{-3}{4} = \frac{x}{4a}$ (h) $\frac{2}{9} = \frac{x}{18a}$

(i) $\frac{-3}{-13} = \frac{x}{26b}$ (j) $\frac{3}{-2b} = \frac{x}{16bc}$

(k) $\frac{0}{100} = \frac{x}{10}$ (l) $4 = \frac{x}{9}$

2. Find and express the sums in lowest terms.

(a) $\frac{4}{5} + \frac{-3}{5}$ (b) $\frac{1}{9} + \frac{3}{9}$

(c) $\frac{6}{a} + \frac{7}{a}$ (d) $\frac{p}{b} + \frac{2a}{b}$

(e) $\frac{7}{8} + \frac{1}{4}$ (f) $\frac{1}{2} + \frac{1}{4}$

(g) $\frac{-3}{7} + \frac{1}{21}$ (h) $\frac{4}{3} + \frac{-1}{9}$

(i) $\frac{-4}{5} + \frac{13}{5}$ (j) $\frac{a}{4} + \frac{c}{8}$

(k) $\frac{2}{-1} + \frac{5}{-1}$ (l) $\frac{4}{7} + \frac{1}{11}$

(m) $\frac{9}{2a} + \frac{3}{4a}$ (n) $\frac{6}{x} + \frac{-3}{ax}$

(o) $\frac{1}{7} + \frac{1}{3}$ (p) $\frac{2}{3} + \frac{3}{7}$

(q) $\frac{4}{x} + \frac{3}{ax}$ (r) $\frac{1}{a} + \frac{1}{b}$

(s) $\frac{3}{x} + \frac{14}{y}$ (t) $\frac{2}{b} + \frac{3}{d}$

3. Complete the following.

(a) $\frac{1}{m} + \frac{-1}{n} = \frac{\blacksquare}{mn} + \frac{\blacksquare}{mn} = \frac{\blacksquare}{mn}$

(b) $\frac{2}{a} + \frac{-3}{x} = \frac{\blacksquare}{ax} + \frac{\blacksquare}{ax} = \frac{\blacksquare}{ax}$

(c) $\frac{4}{b} + \frac{3}{d} = \frac{\blacksquare}{bd} + \frac{\blacksquare}{bd} = \frac{\blacksquare}{bd}$

(d) $\frac{3m}{p} + \frac{-2n}{r} = \frac{\blacksquare}{pr} + \frac{\blacksquare}{pr} = \frac{\blacksquare}{pr}$

(e) $\frac{y}{x} + \frac{z}{s} = \frac{\blacksquare}{xs} + \frac{\blacksquare}{xs} = \frac{\blacksquare}{xs}$

(f) $\frac{7}{6} + \frac{-4}{5} = \frac{\blacksquare}{30} + \frac{\blacksquare}{30} = \frac{\blacksquare}{30}$

4. The length of a box is $4\frac{2}{3}$ inches. The width is $3\frac{1}{2}$ inches. The height is $6\frac{3}{4}$ inches. Find the perimeter of each face. Find the volume of the box.

2·7 Working With Rational Numbers: Subtraction

Example 1

Find the difference in each case.

(a) $\frac{3}{5} - \frac{1}{5}$

(b) $2\frac{5}{7} - \frac{5}{6}$

Solution

(a) $\frac{3}{5} - \frac{1}{5} = \frac{3}{5} + \frac{-1}{5}$

$= \frac{3 + (-1)}{5}$

$= \frac{2}{5}$

(b) $2\frac{5}{7} - \frac{5}{6} = 2\frac{5}{7} + \frac{-5}{6} = \frac{19}{7} + \frac{-5}{6}$

$= \frac{19 \cdot 6}{7 \cdot 6} + \frac{7(-5)}{7 \cdot 6} = \frac{114}{42} + \frac{(-35)}{42}$

$= \frac{79}{42}$

General statement

To subtract any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$

$\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{(-c)}{d}$ Then the operation of addition is performed.

EXERCISE 2·7

1. Complete each statement.

   (a) $\frac{4}{5} - \frac{2}{3} = \frac{4}{5} + \blacksquare$

   (b) $\frac{3}{5} - \frac{1}{3} = \frac{3}{5} + \blacksquare$

   (c) $\frac{4a}{3} - \frac{-2a}{6} = \frac{4a}{3} + \blacksquare$

   (d) $\frac{3}{14} - \frac{-1}{2a} = \frac{3}{14} + \blacksquare$

2. Find these differences and express them in lowest terms.

   (a) $\frac{3}{4} - \frac{1}{4}$ (b) $\frac{7}{8} - \frac{3}{8}$

   (c) $\frac{7}{16} - \frac{-5}{16}$ (d) $\frac{3}{5} - \frac{4}{15}$

   (e) $\frac{4}{a} - \frac{-3}{8a}$ (f) $\frac{4b}{7c} - \frac{-2}{3}$

   (g) $\frac{7x}{2y} - \frac{-3}{4}$ (h) $\frac{9}{3y} - \frac{-3}{4x}$

   (i) $\frac{a}{b} - \frac{c}{d}$ (j) $\frac{9}{5a} - \frac{4}{-6a}$

3. When $a = \frac{1}{3}$, $b = \frac{3}{4}$, $c = \frac{2}{5}$ and $d = \frac{3}{5}$ state, in lowest terms, the rational number represented by each expression.

   $\frac{a}{2}$ $a^2 + 2b$ $3b + 2c + 4d + a$

   $\frac{b}{2c}$ $\frac{ad}{2b}$ $\frac{2a^2b^2}{c^2}$ $\frac{4b + a}{3c + 2d}$

   $\frac{bcd}{abc}$ $\frac{3ab}{cd}$ $b^2 + 4c^2 + 2bc + cd$

4. In each of the following suggest possible values for the variables so that the expression will *not* represent a rational number.

   $\frac{17}{2 - b}$ $\frac{15d}{2d - c}$ $\frac{bc}{d^2}$ $\frac{33}{2m - 3n}$

   $\frac{4}{4x - 2}$ $\frac{5}{4x + 2y - 2}$ $\frac{6}{a^2 + b^2}$

5. Find the following quotients involving mixed numbers.

   $3\frac{1}{3} \div 2\frac{1}{2}$ $4\frac{1}{3} \div 9\frac{1}{4}$ $4\frac{1}{5} \div 2\frac{1}{3}$ $32\frac{1}{3} \div 12\frac{1}{8}$

6. Area of Great Lakes in Canada:

Lake Superior	31,820 square miles
Lake Erie	9,940 square miles
Lake Huron	23,010 square miles
Lake Ontario	7,540 square miles

It is estimated that 125,000,000 tons of sewage are dumped into the lakes annually.

(a) On the average, how many tons per square mile are dumped into the lakes?

(b) An estimate states it will cost $175,000 per square mile to clean up the lakes. What is the cost to clean up the lakes?

(c) If the population of Canada is 21,324,000 what is the cost per Canadian to clean up the lakes?

(d) If the average family in Canada is 4.4 persons, how much per family will it cost to clean up the lakes?

(e) How much would it cost your father if he had to pay for his family? This is the cost of pollution!

2·8 Properties of the Rational Number System

The set of rational numbers has one important property which the set of integers does not have. In the set of integers there is no number a such that

$$4 \times a = 1.$$

We know the number referred to in this instance is $\frac{1}{4}$ but $\frac{1}{4}$ is a member of the set of rational numbers. The multiplicative inverses of -3, $\frac{2}{3}$, $\frac{-5}{9}$ are $\frac{-1}{3}$, $\frac{3}{2}$, $\frac{-9}{5}$ respectively.

Sometimes the inverse of a rational like $\frac{2}{3}$ is written $\frac{1}{\frac{2}{3}}$ but in simplest form it is written $\frac{3}{2}$.

> The product of a number and its multiplicative inverse is 1

> Each rational number a, $a \neq 0$, has a multiplicative inverse, $\frac{1}{a}$

Properties of the set of integers are also properties of the set of rational numbers. The properties of the set of rational numbers are summarized below.

1. $Q = \left\{ \frac{a}{b} \middle| \, a, b \in I, b \neq 0 \right\}$
2. Q is closed under the four basic operations of addition, subtraction, multiplication and division (except for division by 0).
3. Addition and multiplication in Q are associative.
4. Addition and multiplication in Q are commutative.
5. Multiplication is distributive over addition in Q.
6. The multiplicative identity element is 1.
7. The additive identity element is 0.
8. Each element in Q (except 0), possesses a multiplicative inverse.
9. Each element in Q (except 0), possesses an additive inverse.

Students may find properties in 6 through 9 more useful when stated this way:

Addition property of opposites (APO)

For any rational number $\frac{a}{b}$,

$$\frac{a}{b} + \left(-\frac{a}{b}\right) = 0$$

Multiplication property of inverses (MPI)

For any rational number $\frac{a}{b}$, $a \neq 0$,

$$\frac{a}{b} \times \frac{b}{a} = 1$$

Multiplication property of 1 (MP1)

For any rational number $\frac{a}{b}$,

$$\frac{a}{b} \times 1 = \frac{a}{b}$$

Additive property of 0 (APZ)

For any rational number $\frac{a}{b}$,

$$\frac{a}{b} + 0 = \frac{a}{b}$$

Does the word "rational" mean the same here as in mathematics? Explain this "rational".

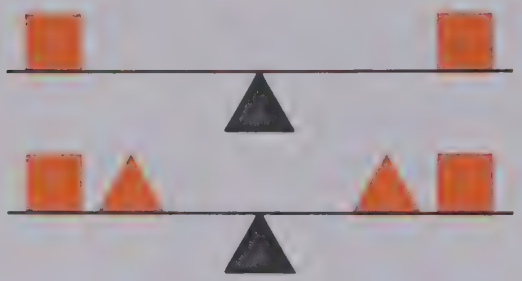
The following are some other properties that you have studied. They may be used when working in the set of rationals.

The Properties of Equality

1. Addition property of equality (APE)

   For any rational numbers $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, if $\frac{a}{b} = \frac{c}{d}$,

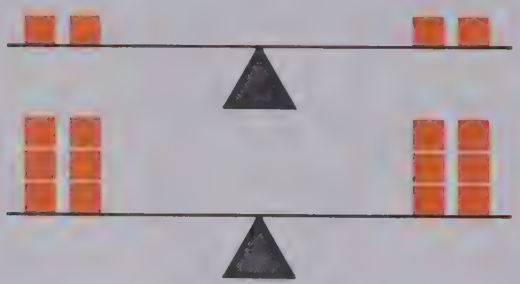
   then $\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}$.



2. Multiplication property of equality (MPE)

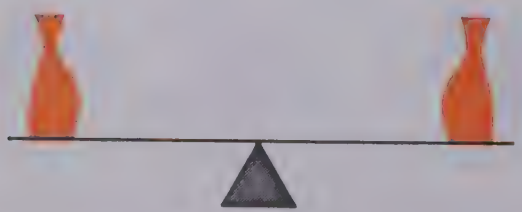
   For any rational numbers $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, if $\frac{a}{b} = \frac{c}{d}$,

   then $\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{e}{f}$.



3. Reflexive property of equality (RPE)

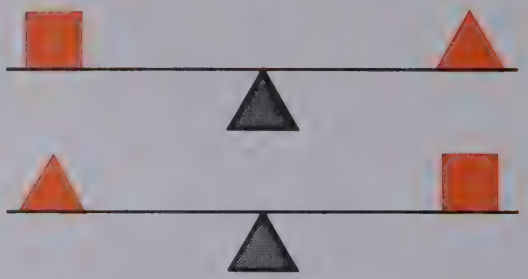
   For any rational number $\frac{a}{b}, \frac{a}{b} = \frac{a}{b}$.



4. Symmetric property of equality (SPE)

   For any rational numbers $\frac{a}{b}, \frac{c}{d}$, if $\frac{a}{b} = \frac{c}{d}$,

   then $\frac{c}{d} = \frac{a}{b}$.



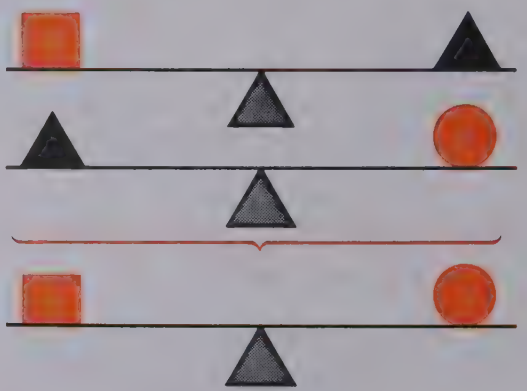
5. Transitive property of equality (TPE)

   For any rational numbers $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$,

   if $\frac{a}{b} = \frac{c}{d}$

   and $\frac{c}{d} = \frac{e}{f}$,

   then $\frac{a}{b} = \frac{e}{f}$.



EXERCISE 2·8

1. Each statement is an instance of a property of the rational numbers. Name the property in each case.

   (a) $(\frac{-3}{4} \times \frac{2}{5})$ is a unique rational number.

   (b) $\frac{4}{3} \div 3$ is a rational number.

(c) $\frac{2}{3}(\frac{4}{5}+3)=\frac{2}{3}\times\frac{4}{5}+\frac{2}{3}\times 3$

(d) $\frac{3}{7}\times 1=1\times\frac{3}{7}=\frac{3}{7}$

(e) $\frac{-6}{11}\times\frac{11}{-6}=1$

(f) $\frac{8}{9}+\frac{-3}{7}=\frac{-3}{7}+\frac{8}{9}$

(g) $\frac{-4}{7}+\frac{4}{7}=0$

(h) $\frac{-5}{11}\times\frac{6}{7}=\frac{6}{7}\times\frac{-5}{11}$

(i) $\frac{49}{67}+0=\frac{49}{67}$

(j) $\frac{4}{3}+\frac{1}{7}$ is a unique rational number.

(k) $\frac{1}{3}-\frac{3}{4}$ is a unique rational number.

2. Indicate by a check those sets which contain a solution for each condition.

Sets of Numbers	*N*	*W*	*I*	*Q*
(a) $a+3=4$				
(b) $x+5=7$				
(c) $4-x=4$				
(d) $2x=0$				
(e) $2x+4=0$				
(f) $3x+5=6$				
(g) $8y-4+y=7$				
(h) $x^2=4$				
(i) $2x^2=18$				
(j) $3x^2=0$				

3. Each of the following is an instance of a property listed in this section. Identify the property, $x, y \in Q$.

(a) If $x+\frac{2}{3}=3$, then $x+\frac{2}{3}-\frac{2}{3}=3-\frac{2}{3}$

(b) If $\frac{8}{9}=x-6$, then $x-6=\frac{8}{9}$

(c) If $\frac{2}{3}x=9$, then $\frac{2}{3}x\times\frac{3}{2}=9\times\frac{3}{2}$

(d) If $\frac{4}{3}\times\frac{3}{4}x=\frac{4}{3}\times 5$, then $x=\frac{4}{3}\times 5$

(e) If $x=\frac{4}{3}$ and $y=\frac{4}{3}$, then $x=y$

(f) If $\frac{4}{3}+(\frac{2}{3}+-\frac{2}{3})=y$, then $\frac{4}{3}=y$

(g) If $\frac{2}{3}+y=x$, then $x=\frac{2}{3}+y$

(h) $\frac{2}{3}\times x\times\frac{3}{2}=\frac{2}{3}\times\frac{3}{2}\times x$

(i) $\frac{5}{7}(x+\frac{2}{3})=\frac{5}{7}x+\frac{5}{7}\times\frac{2}{3}$

(j) If $\frac{7}{9}+x=\frac{1}{3}$, then $x+\frac{7}{9}=\frac{1}{3}$

(k) If $\frac{2}{3}(7+y)=4$,
then $\frac{3}{2}\times\frac{2}{3}(7+y)=\frac{3}{2}\times 4$

(l) If $\frac{3}{2}\times\frac{2}{3}(7+y)=6$, then $7+y=6$

(m) If $7+y=6$, then $7+(-7)+y=6+(-7)$

(n) If $7+(-7)+y=6+(-7)$,
then $y=6+(-7)$

(o) If $y=\frac{-2}{3}$ and $\frac{-2}{3}=x$, then $y=x$

(p) If $\frac{7}{16}-y=x-\frac{1}{2}$,
then $\frac{7}{16}-y+\frac{1}{2}=x-\frac{1}{2}+\frac{1}{2}$

(q) $\frac{4}{9}+(\frac{2}{3}+y)=(\frac{4}{9}+\frac{2}{3})+y$

(r) $\frac{7}{8}\times\frac{8}{7}y=y$

(s) If $\frac{3}{2}y-\frac{2}{3}=\frac{6}{5}$,
then $\frac{3}{2}y-\frac{2}{3}+\frac{2}{3}=\frac{6}{5}+\frac{2}{3}$

(t) If $\frac{3}{2}y-\frac{2}{3}+\frac{2}{3}=\frac{28}{15}$, then $\frac{3}{2}y=\frac{28}{15}$

(u) If $\frac{3}{2}y=\frac{28}{15}$, then $\frac{2}{3}\times\frac{3}{2}y=\frac{2}{3}\times\frac{28}{15}$

(v) If $\frac{2}{3}\times\frac{3}{2}y=\frac{56}{45}$, then $y=\frac{56}{45}$

4. State each of the *Properties of Equality* in your own words—as briefly as you can. Hereafter you may use your own terms for these properties.

5. In January 1971, 1,200,000 8-ounce tins of tuna were taken off the market and destroyed because of mercury contamination. How many tons of tuna were destroyed? A housewife can make a tuna casserole for a family of six from two 8-ounce tins of tuna. How many persons could be given a meal from this much tuna? An 8-ounce tin of tuna sells for 47 cents. How much money was lost by destroying this tuna?

2·9 Working With Properties of Numbers

The simplest equation we can have is one like $x = 3$ or $m = 4$ or $n = -3$. When we have a more complicated equation (or condition) we always want to reduce it to the simplest form. From the simplest form we can name the solution set.

Example 1

Tabulate the solution set for the condition $x + 3 = 6$.

Solution

$$x + 3 = 6$$

What is the additive inverse of 3?
It is -3.
This we add to each side of the condition.

	Reason
$x + 3 + (-3) = 6 + (-3)$	Addition property of equality
$x + 0 = 6 + (-3)$	Addition property of opposites
$x = 6 + (-3)$	Addition property of zero
$x = 3$	Number fact

Solution set is $\{3\}$.

Check

$$\begin{aligned} x + 3 &= 6 \\ L.H.S. &= 3 + 3 \\ &= 6 \\ &= R.H.S. \end{aligned}$$

Example 2

Tabulate the solution set of the condition $3x = 10$.

Solution

$$3x = 10$$

What is the multiplicative inverse of 3?
It is $\frac{1}{3}$.
We multiply each side of the condition by $\frac{1}{3}$.

$\frac{1}{3}(3x) = \frac{1}{3}(10)$	Multiplicative property of equality
$(\frac{1}{3} \times 3)x = \frac{1}{3}(10)$	Associative property of rational numbers
$1x = \frac{1}{3}(10)$	Multiplicative property of inverses
$x = \frac{1}{3}(10)$	Multiplicative property of 1
$x = \frac{10}{3}$	Number fact

Solution set is $\{\frac{10}{3}\}$.

Check

$$\begin{aligned} 3x &= 10 \\ L.H.S. &= 3 \times \tfrac{10}{3} \\ &= 10 \\ &= R.H.S. \end{aligned}$$

Example 3

Tabulate the solution set to the condition $4x + 9 = 7$.

Solution

$$4x + 9 = 7$$

What is the additive inverse of 9?

$4x + 9 + (-9) = 7 + (-9)$	APE
$4x + 0 = 7 + (-9)$	APO
$4x = 7 + (-9)$	APZ
$4x = -2$	Number Fact

What is the multiplicative inverse of 4?

$\frac{1}{4}(4x) = \frac{1}{4}(-2)$	MPE
$(\frac{1}{4} \times 4)x = \frac{1}{4}(-2)$	APM
$1x = -\frac{1}{2}$	Number Fact
$x = -\frac{1}{2}$	MP1

Solution set is $\{-\frac{1}{2}\}$.

Check

$$\begin{aligned} 4x + 9 &= 7 \\ L.H.S. &= 4(-\tfrac{1}{2}) + 9 \\ &= -2 + 9 \\ &= 7 \\ &= R.H.S. \end{aligned}$$

Example 4

Follow carefully each step in this solution.

$$\frac{3x+4}{2}+\frac{2x-1}{3}=4$$

$$6\times\left(\frac{3x+4}{2}+\frac{2x-1}{3}\right)=6\times 4 \qquad \text{MPE (why 6?)}$$

$$6\left(\frac{3x+4}{2}\right)+6\left(\frac{2x-1}{3}\right)=6\times 4 \qquad \left\{\begin{array}{l}\text{Distributive}\\ \text{property}\end{array}\right.$$

$$9x+12+4x-2=24 \qquad \left\{\begin{array}{l}\text{Number facts and}\\ \text{distributive prop.}\end{array}\right.$$

$$9x+4x+12-2=24 \qquad \text{Commutative prop.}$$

$$(9+4)x+12-2=24 \qquad \text{Distributive prop.}$$

$$13x+10=24 \qquad \text{Number facts}$$

$$13x+10-10=24-10 \qquad \text{APE}$$

$$13x+0=14 \qquad \text{Number facts}$$

$$13x=14 \qquad \text{APZ}$$

$$\tfrac{1}{13}\times 13x=\tfrac{1}{13}\times 14 \qquad \text{MPE}$$

$$x=\tfrac{14}{13}$$

Solution set is $\{\frac{14}{13}\}$.

Check

$$\text{L.H.S.}=\frac{3\times\frac{14}{13}+4}{2}+\frac{2\times\frac{14}{13}-1}{3}$$

$$=\frac{\frac{42}{13}+\frac{52}{13}}{2}+\frac{\frac{28}{13}-\frac{13}{13}}{3}$$

$$=\frac{\frac{94}{13}}{2}+\frac{\frac{15}{13}}{3}$$

$$=\tfrac{94}{13}\times\tfrac{1}{2}+\tfrac{15}{13}\times\tfrac{1}{3}$$

$$=\tfrac{47}{13}+\tfrac{5}{13}$$

$$=\tfrac{52}{13}$$

$$=4$$

$$=\text{R.H.S.}$$

EXERCISE 2·9

1. Tabulate the solution set for each. Indicate the property used to find each equivalent condition. Check each.

(a) $4n=16$ (b) $5n=22$

(c) $-3n=14$ (d) $x+4=5$

(e) $x-4=6$ (f) $x+17=-15$

(g) $3x+4=5$ (h) $2x-4=8$

(i) $\dfrac{5x-4}{2}=-12$ (j) $4-3x=5$

(k) $3=4x$ (l) $\dfrac{4-5x}{3}=12$

(m) $\frac{3}{4}x+\frac{1}{2}x+4=\frac{3}{4}$ (n) $\frac{3}{4}=\frac{2}{5}x+4$

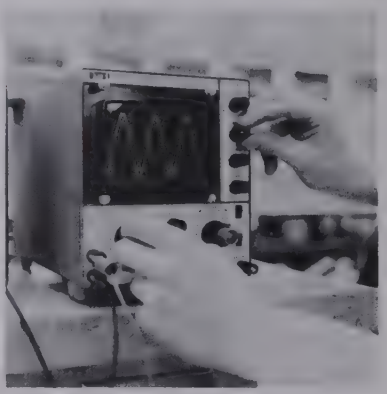
(o) $\dfrac{3x+4}{2}+\dfrac{x-1}{3}=4$ (p) $\frac{4}{9}x-\frac{2}{3}=\frac{1}{3}$

2. In takeoff a jet plane pours 21 pounds of pollutants per jet engine into the air. If, at the Toronto airport, each of three major airlines averages 114 flights of 4-engine jet planes per day, how many tons of pollutant are poured into the air daily? weekly? yearly?

2·10 Periodic Decimals

Rational numbers are not always represented in the form $\frac{a}{b}$.

Rational numbers may be represented in decimal form. We now show how a rational number of the form $\frac{a}{b}$ is converted to its equivalent decimal form.



How might these people encounter periodic decimals?

Example 1

Convert each rational number to its equivalent decimal form:

(a) $\frac{13}{16}$

(b) $\frac{11}{8}$

Solution

(a)
$$\begin{array}{r} .8125000\ldots \\ 16\,)\overline{13.0000} \\ \underline{128} \\ 20 \\ \underline{16} \\ 40 \\ \underline{32} \\ 80 \\ \underline{80} \\ 00 \\ \underline{0} \\ 0 \end{array}$$

Therefore, $\frac{13}{16} = 0.81250$ or 0.8125

(b)
$$\begin{array}{r} 1.375 \\ 8\,)\overline{11.000} \\ \underline{8} \\ 30 \\ \underline{24} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

Therefore, $\frac{11}{8} = 1.375$

Notice in Example 1 the denominator is divided into the numerator and the remainder in each step of the division is smaller than the divisor, 16. Notice that when the remainder is zero there are only zeros in the quotient thereafter. It is customary not to divide any more once a zero appears as a remainder. But you will notice that the rational number $\frac{13}{16}$ can be expressed as a decimal in which a part of the quotient – the zero – repeats.

$$\frac{13}{16} = .8125000\ldots$$

Unless there is a need for the zeros in the quotient, it is customary to stop the division when a zero appears in the remainder. See Example 1(b). We must realize though that this is a repeating decimal but the part that repeats is omitted. (These are often called terminating decimals.)

Not all rational numbers when written in their decimal forms have zeros for their repeating part Sometimes a single digit repeats and sometimes a group of several digits repeats. Study Example 2.

Example 2

Find a decimal numeral to express the rational numbers:

(a) $\frac{4}{3}$

(b) $\frac{43}{7}$

Solution

(a)
$$\begin{array}{r} 1.333\ldots \\ 3\overline{)\,4.000} \\ 3 \\ \hline 10 \\ 9 \\ \hline 10 \\ 9 \\ \hline 10 \end{array}$$

Starts to repeat

Therefore $\frac{4}{3} = 1.\dot{3}$

The dot over the 3 means the 3 repeats.

(b)
$$\begin{array}{r} 6.142857\ldots \\ 7\overline{)\,43.000} \\ 42 \\ \hline 10 \\ 7 \\ \hline 30 \\ 28 \\ \hline 20 \\ 14 \\ \hline 60 \\ 56 \\ \hline 40 \\ 35 \\ \hline 50 \\ 49 \\ \hline 10 \end{array}$$

Starts to repeat

Therefore $\frac{43}{7} = 6.\dot{1}42857\dot{}$

The dots over the 1 and 7 mean that these and all the digits in between are repeated in that order.

All rational numbers can be written as repeating decimal numerals. The interval or part that repeats is called the *repetend*. In Example 2(a) the repetend is 3, while in (b) the repetend is 142857. There are several ways of showing that the numeral group repeats.

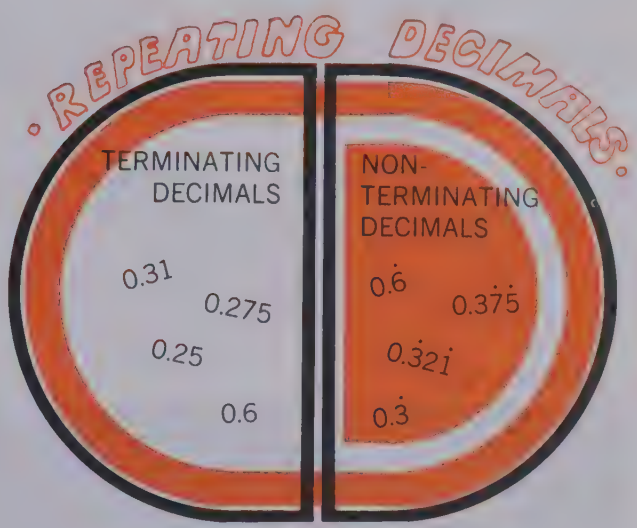
$$\left.\begin{array}{l} 6.\dot{1}4\dot{2}8\dot{5}7\dot{} \\ 6.\overline{142857} \\ 6.\dot{1}42857\dot{} \end{array}\right\} = 6.142857142857\ldots$$

We shall use the last method.

In summary, rational numbers can be written as repeating decimals. For convenience, we will call those repeating decimal numerals which have only zero as their repetend *terminating decimals*.

Those decimal numerals whose repetends are not just zeros will be called *nonterminating decimals*.

The terminating and nonterminating decimals are both subsets of the set of repeating decimals.



EXERCISE 2·10

Calculators, if available, should be used where appropriate in the balance of this unit.

Write each of the following as a decimal numeral.

1. $\frac{4}{5}$ 2. $\frac{1}{4}$ 3. $\frac{1}{8}$ 4. $\frac{1}{16}$

5. $\frac{5}{16}$ 6. $\frac{3}{32}$ 7. $\frac{75}{64}$ 8. $\frac{3}{128}$

Write the equivalent decimal number for each of the following.

9. $\frac{1}{3}$ 10. $\frac{1}{6}$ 11. $\frac{1}{9}$ 12. $\frac{2}{3}$

13. $\frac{5}{6}$ 14. $\frac{4}{9}$ 15. $\frac{19}{32}$ 16. $\frac{51}{32}$

17. $\frac{5}{11}$ 18. $\frac{1}{13}$ 19. $\frac{2}{13}$ 20. $\frac{1}{99}$

21. Write the decimal numerals for $\frac{1}{7}$, $\frac{2}{7}$, and $\frac{3}{7}$. State the repetend for each. Try to find a short cut method of calculating the decimal numerals for $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$.

Find decimal numerals for each of the following. What is the repetend for each? What do the repetends have in common? Compare the denominator of these numbers with the denominators in question 21. What is there in common?

22. $\frac{1}{21}$ 23. $\frac{23}{14}$ 24. $\frac{17}{35}$ 25. $\frac{43}{42}$

26. Transform $\frac{25}{7}$ to a decimal notation. Do this in the fewest steps possible. See question 21.

27. Each Canadian produces approximately 1,217 pounds of garbage a year. How many pounds per day is this on the average? It is estimated that Europeans average about one-third this. How much garbage does each European produce annually? Daily? Can you suggest reasons for the difference?

2·11 Periodic Decimals Continued

We shall be discussing nonterminating decimals in this section.

Refer to question 21 in Exercise 2·10. In this case you were dividing by 7. What is the smallest remainder possible? What is the largest remainder possible? How many different remainders are possible? If a natural number is divided by a natural number n, what is the smallest remainder possible? What is the largest remainder possible? How many different remainders are possible? If there is only one remainder possible, how many different digits are possible in the repetend? If there are two different remainders possible, how many digits are possible in the repetend? If there are three different remainders possible, how many digits are possible in the repetend? If there are $n - 1$ different remainders possible how many digits are possible in the repetend?

If a natural number p is divided by a natural number n how many digits are possible in the repetend?

The Parthenon at Athens fits into a Golden Rectangle. What are the dimensions of a Golden Rectangle?

EXERCISE 2·11

1. In each of the indicated divisions, how many different remainders are possible?

$\frac{4}{11}$ $\frac{3}{17}$ $\frac{15}{13}$ $\frac{53}{29}$

2. State the maximum number of steps in the division indicated before repetition is bound to happen.

$\frac{6}{7}$ $\frac{23}{13}$ $\frac{41}{19}$ $\frac{31}{23}$

3. State the maximum number of digits possible in the repetend of the decimal numeral that represents each of the rational numbers.

$\frac{3}{7}$ $\frac{4}{13}$ $\frac{20}{17}$ $\frac{31}{19}$ $\frac{141}{47}$

SPECIAL CHALLENGE!

4. Find the decimal numerals for each of these:

$$\frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \cdots, \frac{10}{11}$$

Can you find the patterns in the repetends? How many digits are there in any one repetend?
How many patterns are there in the 10 decimal numerals?

5. Now find the decimal numerals for:

$$\frac{1}{13}, \frac{2}{13}, \frac{3}{13}, \cdots, \frac{12}{13}$$

What is the pattern in the repetends? How many digits are there in any one repetend?
How many patterns are there?

6. How many patterns are there in the set of decimal numerals for the sevenths?

7. Complete this table.

N – the prime number in the denominator
D – the number of digits in any one repetend
P – the number of patterns

N	*D*	*P*
3		
7		
11		
13		
17		
19		
23		

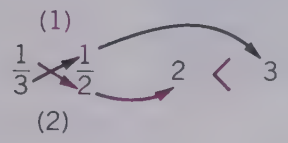
Do you see the relationship?
Now try 47! 97!

2·12 Which Number Is Greater?

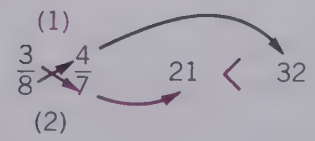
Which number is greater: $\frac{1}{3}$ or $\frac{1}{2}$?
Certainly, you know $\frac{1}{2}$ is the greater – hence, we may write $\frac{1}{3} < \frac{1}{2}$.

Which is greater: $\frac{3}{8}$ or $\frac{4}{7}$? Again, you guessed correctly: $\frac{4}{7}$. But can you tell readily which of this pair is greater: $\frac{4}{13}, \frac{5}{16}$?

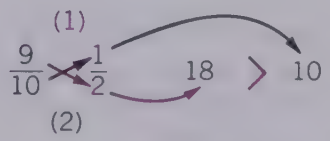
In order to attempt more difficult questions, let us look again at $\frac{1}{3}$ and $\frac{1}{2}$.



If we *cross multiply* in the order shown and record our products as shown we might predict that since $2 < 3$ and $\frac{1}{3} < \frac{1}{2}$, this procedure works. Let us see if it works with $\frac{3}{8}$ and $\frac{4}{7}$.



We know $21 < 32$ and we know $\frac{3}{8} < \frac{4}{7}$. It seems to work. Let us try $\frac{9}{10}$ and $\frac{1}{2}$.



We know $\frac{9}{10}$ is larger than $\frac{1}{2}$ and it turns out that $18 > 10$. It does seem to work.

Use the method just shown to decide which of each pair is the larger:

$\frac{1}{3} ? \frac{7}{8}$ $\frac{6}{7} ? \frac{8}{9}$ $\frac{1}{4} ? \frac{3}{8}$ $\frac{3}{2} ? \frac{5}{4}$ $\frac{3}{6} ? \frac{7}{14}$

Does the method work? What conclusion did you reach in regard to $\frac{3}{6}$ and $\frac{7}{14}$?

Let us generalize the method:

Given any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$

$\frac{a}{b} < \frac{c}{d}$ if $ad < bc$

$\frac{a}{b} > \frac{c}{d}$ if $ad > bc$

$\frac{a}{b} = \frac{c}{d}$ if $ad = bc$

EXERCISE 2·12

1. Arrange the numbers in each group in order of size from smallest to largest.

   (a) $\frac{2}{3}, \frac{5}{7}$ (b) $\frac{5}{11}, \frac{4}{9}$

   (c) $\frac{33}{65}, \frac{11}{15}$ (d) $\frac{14}{71}, \frac{41}{53}$

   (e) $\frac{3}{7}, \frac{10}{22}, \frac{13}{29}$ (f) $\frac{21}{44}, \frac{32}{65}$

2. If each Canadian produces 4 cubic feet of garbage weekly, calculate the volume of garbage produced by the city of Saskatoon with a population of 225,000.

   Mount Blackstrap was created near Saskatoon in 1970 for the Winter Games in 1971. For calculations let us assume it is approximately a pyramid with a square base of 1800 feet and an altitude of 300 feet. How many years of collecting garbage from Saskatoon would it take to produce a pile the size of Mount Blackstrap?

   Hint!

   Volume of pyramid

   $\frac{1}{3}$ *of cuboid with equal dimensions.*

Many cities are experiencing difficulty in disposing of their garbage. Do you have suggestions of what should be done with city garbage?

2·13 Square Root

We now look for a set of numbers that will allow a nonempty solution set for problems such as

$$\{x|x^2 = 2\} \quad \{x|x^2 = 5\}.$$

Let us review briefly the ways in which one can find the square root of a number. If you are asked,

What is the square root of 4?

you think,

What number multiplied by itself gives 4?

You think

$$2 \times 2 = 4$$

and conclude that 2 is the square root of 4.

You also think

$$(-2) \times (-2) = 4$$

and conclude correctly that -2 is also a square root of 4. Therefore you say that the square root of 4 is 2 or -2. In algebra we might write the problem in two ways.

$$\{x|x^2 = 4\} \text{ or } \{x|x = \sqrt{4}\}$$

The solution set is $\{-2, +2\}$

Blackstrap Mountain Ski Lift, Saskatchewan.

What is the solution set for $\{x|x^2 = 81\}$?

Realizing that

$$9 \times 9 = 81 \text{ and } (-9) \times (-9) = 81,$$

you may write the answer $\{-9, 9\}$.

Any positive number has two square roots that differ only in sign. The positive square root is called the *principal square root*. The principal square root of 16 is written

$\sqrt{16}$ — *radical sign*, *radicand*

"$\sqrt{49} = 7$" is read

The principal square root of 49 is 7,

whereas "$-\sqrt{49} = -7$" is read

The negative square root of 49 is negative 7.

If we ask for *the square root of 49* then the solution set is $\{-7, 7\}$, but if we ask for the *principal square root* then the solution set is $\{7\}$. Similarly if we ask for the *negative square root of 49* the solution set is $\{-7\}$.

The principal square roots of the first 15 perfect squares are easy to remember.

n	1	2	3	4	5	6	7	8	9
n^2	1	4	9	16	25	36	49	64	81

n	10	11	12	13	14	15
n^2	100	121	144	169	196	225

2·14 Calculating Square Roots

Example 1

Find the positive square root of 6 to 4 decimal places by Newton's Method.

Solution

$2 < \sqrt{6} < 3$ (since $2^2 = 4$ and $3^2 = 9$)

Our first approximation is 2.5.

Divide 6 by 2.5. Our new approximation is

$$\frac{2.5 + 2.40}{2} = 2.45$$

Divide 6 by 2.45. Our new approximation is the average:

$$\frac{2.45 + 2.4489}{2} = 2.44945.$$

Sir Isaac Newton. (1642-1727) English natural philosopher and mathematician.

EXERCISE 2·14

Which of the following decimals are repeating decimals?

1. 0.4848488484...
2. 0.345763457634576...
3. 0.1234513245...
4. 0.24935793579357...
5. 0.100110011100...
6. 0.3125976312956731...

Use the method of successive approximation (Newton's Method) to find the square roots to the nearest thousandth. Use desk calculators if they are available. (15-18 are optional.)

7. 147
8. 32.4
9. 490
10. 1000
11. 3.47
12. 37.4
13. 374
14. 3740
15. 0.374
16. 0.0374
17. 0.00374
18. 0.000374

19. Find the cube for each number named:
    (a) 1 (b) 2 (c) 3 (d) 4
    (e) 5 (f) 10 (g) −3 (h) −2.

20. Using the calculator and the method of estimating and trial, find the fourth power of each number named in question 19.

21. Remember that *finding the root of* and *finding the power of* are inverse operations. Find the following roots.

(a) $\sqrt[3]{1}$ $\sqrt[3]{8}$ $\sqrt[3]{-1}$
$\sqrt[3]{-8}$ $\sqrt[3]{64}$ $\sqrt[3]{-27}$

(b) $\sqrt[4]{1}$ $\sqrt[4]{16}$ $\sqrt[4]{81}$
$\sqrt[4]{625}$ $\sqrt[4]{10{,}000}$ $\sqrt[4]{160{,}000}$

(c) $\sqrt[5]{-1}$ $\sqrt[5]{1}$ $\sqrt[5]{32}$
$\sqrt[5]{-32}$ $\sqrt[5]{3125}$ $\sqrt[5]{3{,}200{,}000}$

22. Are the numerals in question 21 names for rational or irrational numbers? Why?

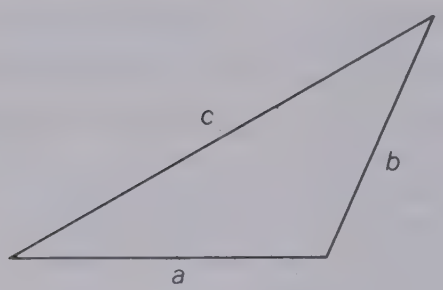
23. In each, name the two consecutive natural numbers between which the number lies.

(a) $\sqrt[3]{50}$ (b) $\sqrt[4]{300}$ (c) $\sqrt[5]{-300}$
(d) $\sqrt[3]{900}$ (e) $\sqrt[4]{9000}$ (f) $\sqrt[5]{90000}$
(g) $\sqrt[4]{4000}$ (h) $\sqrt{320}$ (i) $\sqrt{3200}$

24. The formula for the area of a scalene triangle is

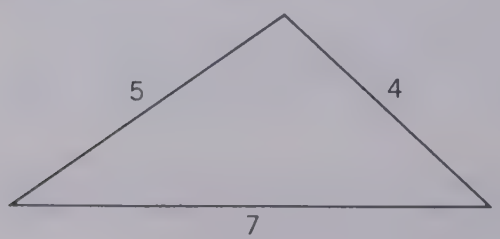
$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$.



Find the area of the triangle when
(a) $a = 6,\ b = 7,\ c = 9$
(b) $a = 8,\ b = 12,\ c = 14$

25. Find the area of the triangle.



26. The amount of heat produced by an electric heat coil is determined by the formula $H = 0.24C^2RT$ where H is the number of calories produced, C is the current in amperes, R is the resistance of the heating element in ohms and T is the time in seconds. A certain coffee percolator has a 50 ohm heating coil and can carry a current of 10 amperes.

(a) How much heat is produced in 1 hour?

(b) How many minutes are required to produce 30,000 calories?

27. An electric heater with a resistance of 60 ohms is used to produce 120,000 calories in 5 minutes. What current will be required?

28. Two wires of equal resistance carry current for the same length of time. The current in the first is one and a half times that in the second. Compare the amounts of heat generated in the two wires.

Interpret this cartoon.

2·15 Nonperiodic Decimals

If we try to calculate the exact square root of 3 we find the operation of taking successive approximations goes on indefinitely. Each successive approximation when squared will give a number close to 3 but the product will never be exactly 3. Successive approximations for $\sqrt{3}$ are 1.5, 1.75, 1.7321, 1.732050 and 1.73205023. Find $(1.73205023)^2$. Does it equal 3 exactly? The student will witness there is no repetition in the decimal. If we continue the approximation for 100 decimal places there still will be no repetition. $\sqrt{3}$ cannot be represented exactly by a repeating decimal. There is a proof that $\sqrt{3}$ is not rational but we shall not study the proof here. The following numbers cannot be represented by repeating decimals:

$$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}, \sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}, \sqrt{17}, \ldots$$

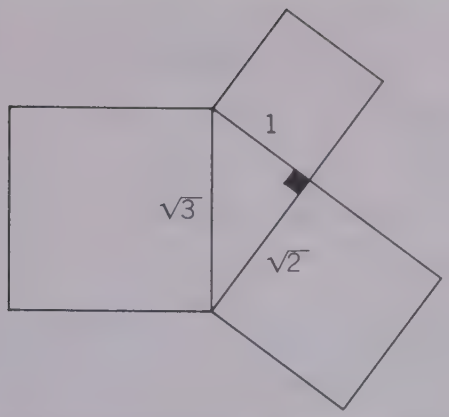
Notice which roots are left out of the sequence:

$$\sqrt{1}, \sqrt{4}, \sqrt{9}, \text{ and } \sqrt{16}.$$

Why are they left out?
What are the next two that will be left out?

The set of repeating decimals is called the rational numbers. The set of nonrepeating decimals is called the *irrationals*. The prefix *ir* means *not*, therefore irrational means *not rational*. The set of irrationals is represented by $\bar{Q}$, read "bar Q".

$\bar{Q} = \{x | x \text{ is a nonrepeating decimal}\}$



There are other nonrepeating decimals besides the roots of numbers which are not perfect squares. Study these nonrepeating decimals.

(a) 0.12112111211112...
(b) 0.12122122212222...
(c) 0.125763141352894765119...
(d) 0.47654765547655...
(e) 0.12123123412345...

There is an infinite number of nonrepeating decimals. A nonrepeating decimal that you use often is π. You can represent the number π by a point in the number line.

Many people are unaware of nonperiodic decimals. Yet they are present all around us. How many items in this picture point to nonperiodic decimals?

Projects

Project 3

Purpose

To locate the unique point π in the number line.

Method

Obtain a vegetable can with diameter about 2 inches. On a line mark off 4 units – each unit being the diameter of the can. Using a soft fine wire (or good quality string) wrap 1 turn about the can cutting the wire so as the two ends *just* meet. Lay this wire out on the line marked with the units. See Figure 2-1. The point on the line coinciding with B represents the number π on the number line.

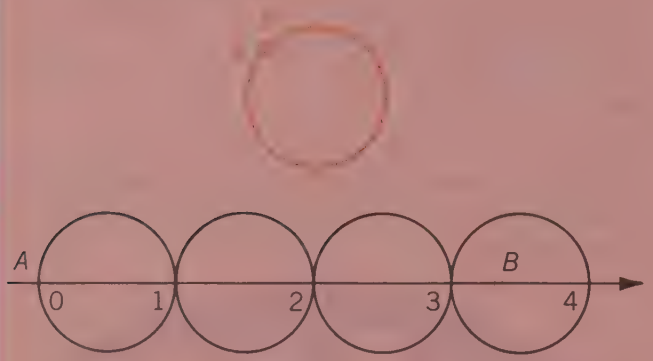


Figure 2-1

We use $\frac{22}{7}$ and 3.14 as approximations for π. Mathematicians have used the computer to find better approximations for π. One approximation to 20 decimal places is

$$\pi = 3.14159265358979323846$$

This is not an exact value. Less than one hundred years ago it was proved that π is irrational.

Project 4

Calculate π from the measurements of a number of different tin cans.

circumference				
diameter				
c/d				

Devise ways of improving accuracy of measurements:

(a) 12 loops of wire produce a more accurate measurement of circumference than just 1

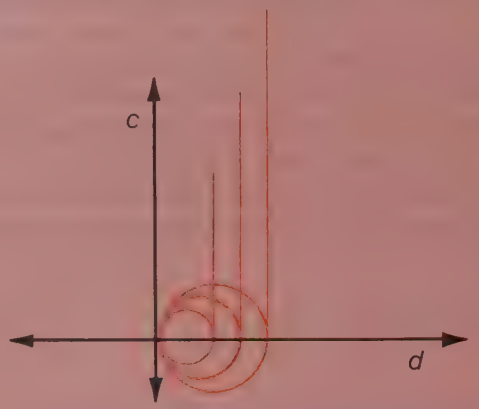
(b) perhaps a paper strip wrapped about the cylinder then pierced with a pin

(c) use of calipers for diameter

(d) working in groups of two's or three's to get 10 or 15 sets of measurements and average them

(e) others.

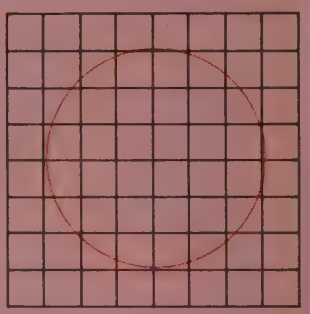
Plot a graph of diameter against circumference. On the same axis plot the relation $c = 3d$.



Project 5

Use graph paper. Draw a circle with some convenient radius. Count squares to find area. Then calculate π from

$$\pi = \frac{A}{r^2}$$



How do you count the squares accurately?

EXERCISE 2·15

1. Refer to the examples of nonrepeating decimals on page 43. Continue the pattern established in (a), (b), (d), (e) by writing 12 to 15 more digits for each.

2. Give 3 examples of nonrepeating decimals. Do not use any given in this section.

2·16 Constructions for Locating Irrational Numbers on the Number Line

Now that we have located π on the number line, we will show how $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ can be located on the number line.

Refer to the Figures 2-2 and 2-3.

On the number line $\overleftrightarrow{AD}$ if point A represents 0, point B represents 1 then point D represents the irrational number $\sqrt{2}$.

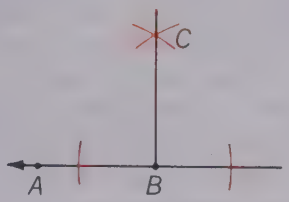


Figure 2-2

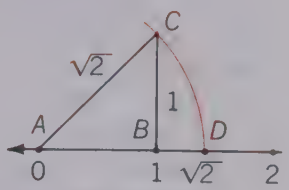


Figure 2-3

Proof

Using the Theorem of Pythagoras we know:

$m(\overline{AC})^2 = m(\overline{AB})^2 + m(\overline{BC})^2$

$m(\overline{AC})^2 = 1 + 1$

$m(\overline{AC}) = \sqrt{2}$.

Study the diagrams in Figures 2-4 to 2-6.

Can you see why the points on the number line are $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{6}$, respectively. Can you think of another way to locate $\sqrt{6}$ on the number line?

Did you notice that $(\sqrt{2})^2 = 2$?

What is $(\sqrt{3})^2$? $(\sqrt{5})^2$? $(\sqrt{6})^2$?

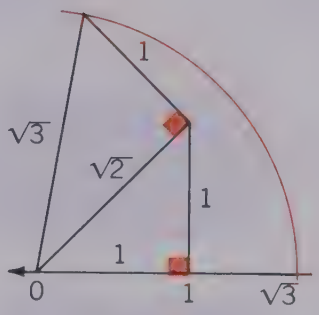


Figure 2-4

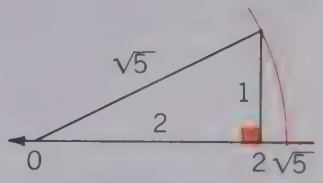


Figure 2-5

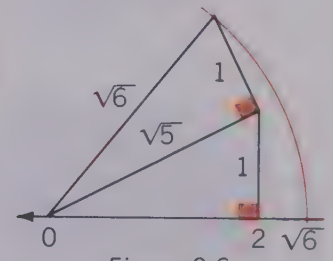


Figure 2-6

EXERCISE 2·16

1. Using compasses, straightedge and ruler construct segments with the following measures. Use the method illustrated in this section.

   (a) $\sqrt{3}$ (b) $\sqrt{7}$ (c) $\sqrt{10}$

   (d) $\sqrt{17}$ (e) $\sqrt{18}$ (f) $\sqrt{26}$

2. The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Use a calculator to find the volume of a sphere whose radius is 8 when

   (a) $\pi = 3.1416$ (b) $\pi = 3.142$ (c) $\pi = 3.14$.

   Do you think 3.14 is an accurate enough value for π for most physical measurements? Explain.

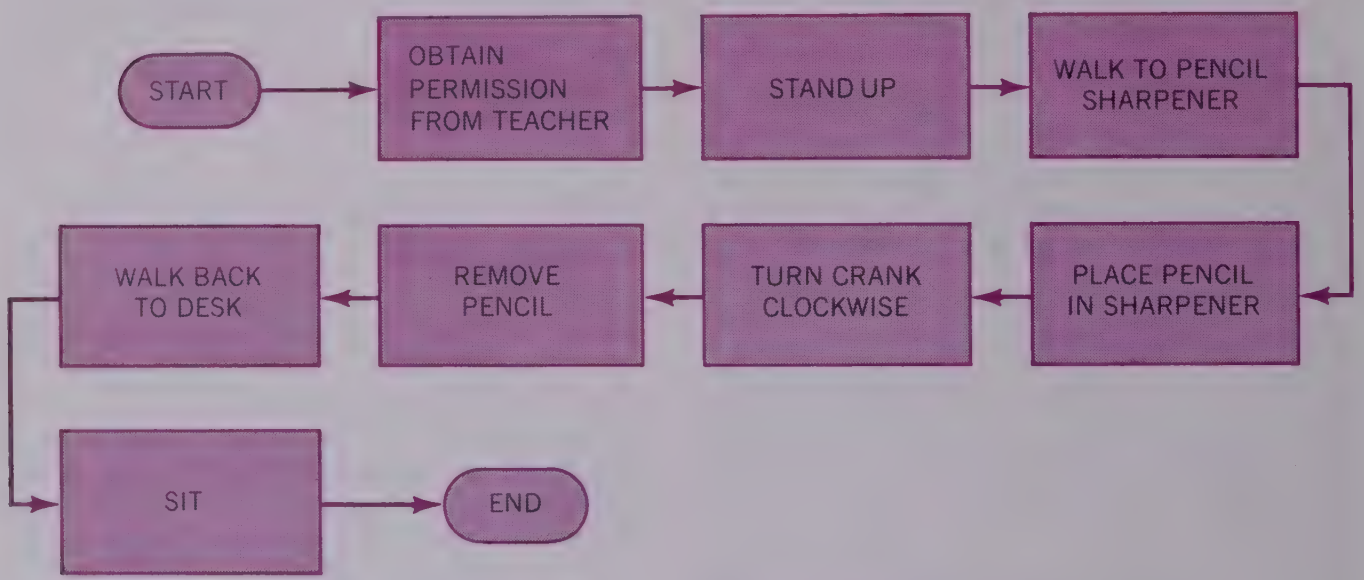
3. The population of Winnipeg is approximately 764,000. It is estimated that each resident produces 5.5 cubic feet of garbage a week. If the city decided to treat the garbage, then stockpile it to build a cone-shape mountain for skiing, how long would it take to build a mountain 400 feet high with a radius of 1200 feet? (Volume of cone $= \frac{1}{3}\pi r^2 h$.) What would be the length of the slant height of the cone?

2·17 Charting Your Work

Have you ever wondered how mathematicians and scientists use computers to solve problems and to do the many things computers can do? The mathematician must translate the problem into a suitable form, then *feed* the data into the computer with appropriate instructions. With the advent of computers, *flow charts* came into widespread use. A flow chart is a chart consisting of labelled boxes and arrows showing the sequence of the operations and the relationships among the data and the logical steps. Study the simplified flow chart that shows how to sharpen a pencil.

Let us write the steps for a simplified flow chart for getting a drink of water.

1. Am I thirsty?
2. No – Wait awhile. Go to step 1.
3. Yes – Is someone at water fountain?
4. Yes – Wait awhile. Go to step 3.
5. No – Raise hand. Ask for permission.
6. Is permission granted?
7. No – Wait awhile. Go to step 3.
8. Yes – Stand up.
9. Walk to fountain and turn on tap.
10. Bend over and drink.
11. Stop drinking and stand up.
12. Is thirst quenched?
13. No – Go to step 10.
14. Yes – Return to work



This is a very simplified diagram. You did not have to make any decisions in following it through. Study the next flow chart (page 47) to see that decision-making questions are stated in a form that requires a YES or NO answer.

Did you notice when each type of box is used? When is a diamond used? a rectangle? a loop?

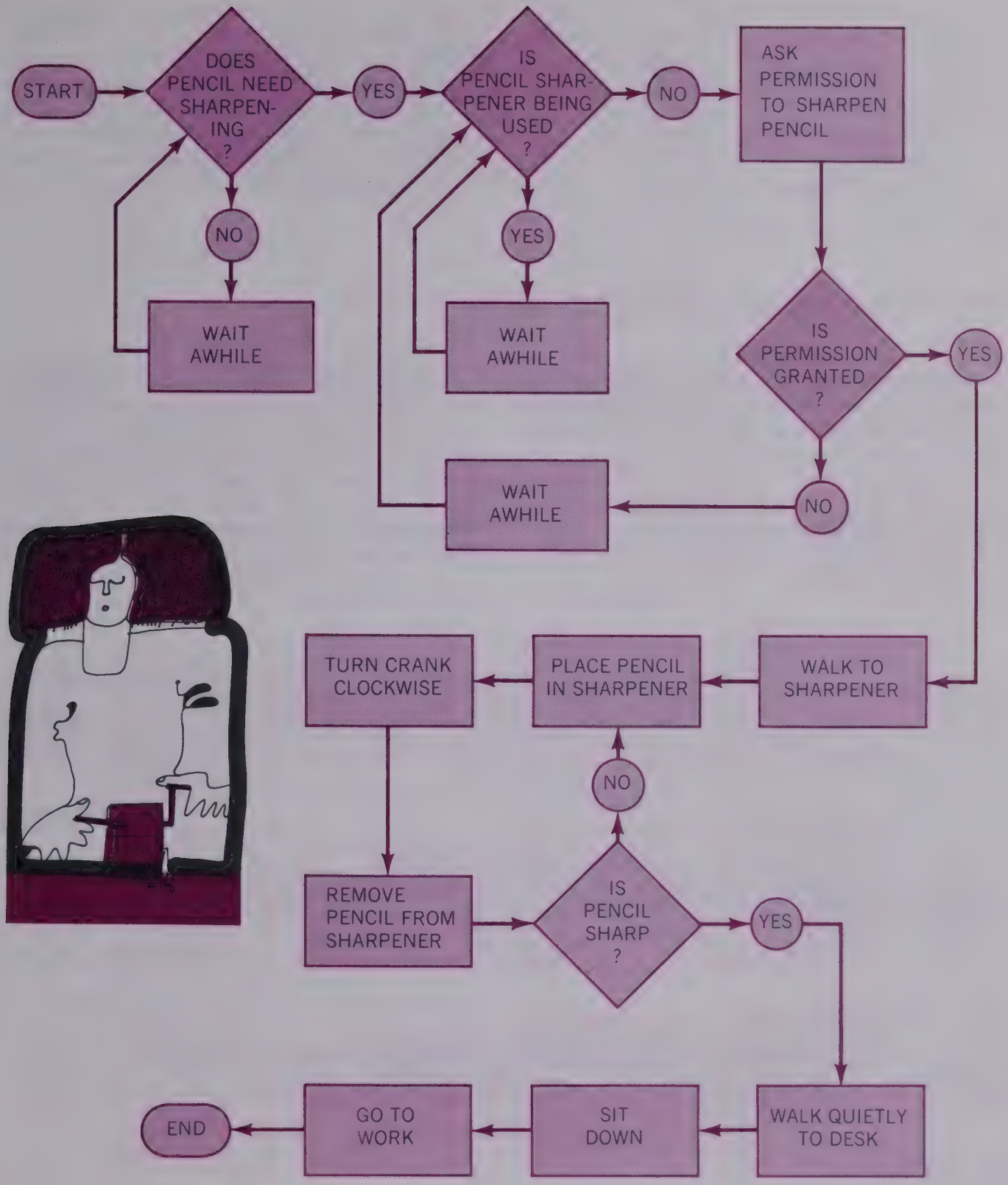
Before you attempt to construct a flow chart, it is wise to put down in step form the sequential actions necessary and the questions that may arise.

EXERCISE 2·17

1. Place the steps above for getting a drink of water in a flow chart using the proper shapes and arrows.

2. List the steps for each:
   (a) How to open a door.
   (b) How to brush your teeth.
   (c) How to get a book from the library.

3. Place the steps of each part in question 2 in a flow chart.


START
DOES PENCIL NEED SHARPEN-ING ?
YES
NO
WAIT AWHILE
IS PENCIL SHAR-PENER BEING USED ?
YES
NO
WAIT AWHILE
ASK PERMISSION TO SHARPEN PENCIL
IS PERMISSION GRANTED ?
YES
NO
WAIT AWHILE
WALK TO SHARPENER
PLACE PENCIL IN SHARPENER
TURN CRANK CLOCKWISE
NO
IS PENCIL SHARP ?
YES
REMOVE PENCIL FROM SHARPENER
WALK QUIETLY TO DESK
SIT DOWN
GO TO WORK
END

4. Construct flow charts for several of the following:
   (a) How to take curlers out of your hair (*girls only?*).
   (b) How to back a car out of a garage.
   (c) How to make a fried egg sandwich.
   (d) How to change a flat tire on a car.
   (e) How to put on finger nail polish.
   (f) How to pass a car on the highway.
   (g) How to *shift* gears on a motorbike.
   (h) How to *pass* an examination.
   (i) How to get a date.
5. Make a flow chart for *How to steal a car.* Put in all the decisions one has to make. Does your flow chart indicate under what conditions it is easy to steal a car and under what conditions it is difficult? What does this chart tell you about the care you should exercise when you park your car?

2·18 Using the Slide Rule for $\sqrt{a}$ and $\sqrt[3]{a}$

In **Book 1** you learned to use tables to find decimal approximations of square roots. We will now learn how to use the slide rule to find decimal approximations and to make simple computations.

Example 1

Calculate (a) $\sqrt{4}$ (b) $\sqrt{2}$ (c) $\sqrt{25}$ (d) $\sqrt{132}$

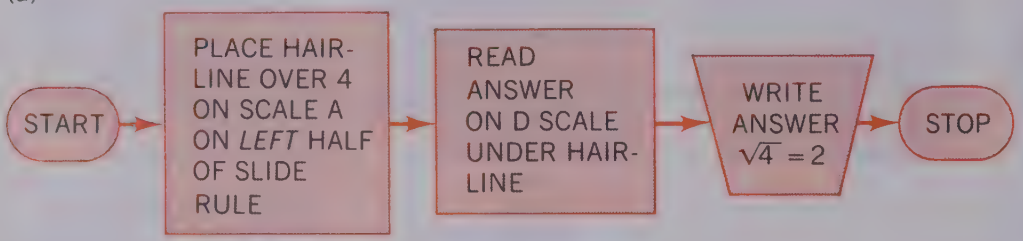
Solution

Use the left half of the A scale to find the square root of a number with an odd number of digits ($\sqrt{2}$, $\sqrt{144}$, $\sqrt{14544}$, etc.).

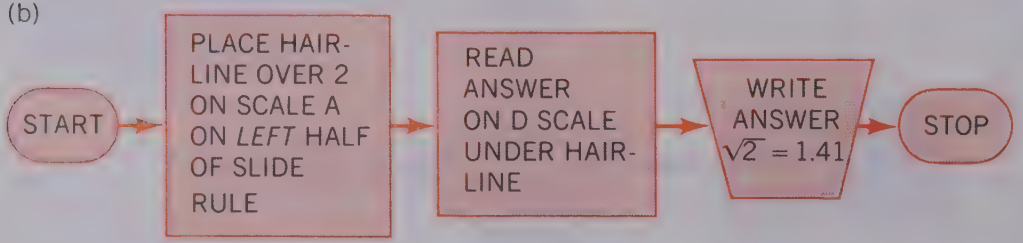
Use the right half of scale A to find the square root of a number with an even number of digits ($\sqrt{16}$, $\sqrt{2500}$, etc.).

Always estimate your answer so that you will know where the decimal is to be placed.

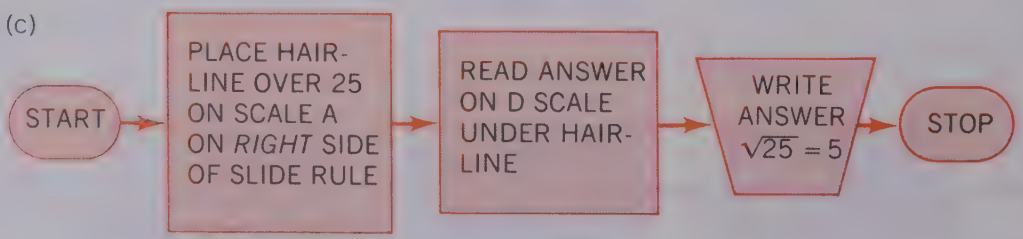
(a)

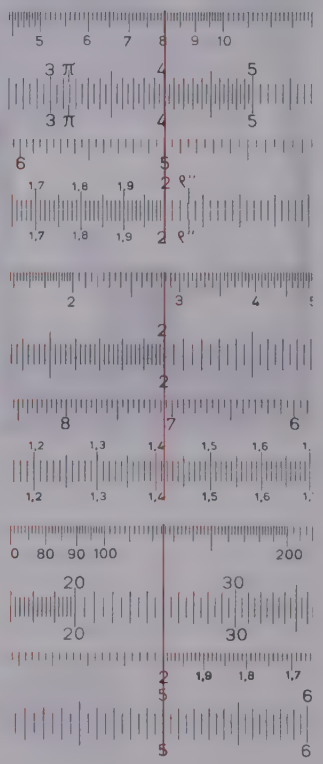

START
PLACE HAIR-LINE OVER 4 ON SCALE A ON *LEFT* HALF OF SLIDE RULE
READ ANSWER ON D SCALE UNDER HAIR-LINE
WRITE ANSWER $\sqrt{4} = 2$
STOP

(b)

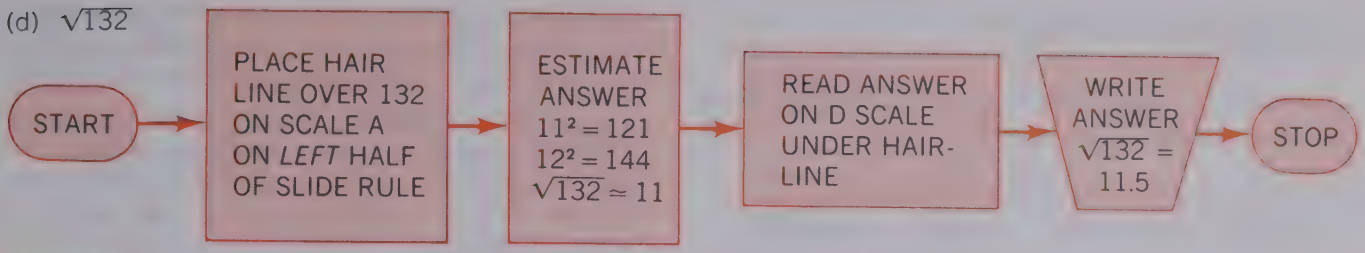

START
PLACE HAIR-LINE OVER 2 ON SCALE A ON *LEFT* HALF OF SLIDE RULE
READ ANSWER ON D SCALE UNDER HAIR-LINE
WRITE ANSWER $\sqrt{2} = 1.41$
STOP

(c)


START
PLACE HAIR-LINE OVER 25 ON SCALE A ON *RIGHT* SIDE OF SLIDE RULE
READ ANSWER ON D SCALE UNDER HAIR-LINE
WRITE ANSWER $\sqrt{25} = 5$
STOP



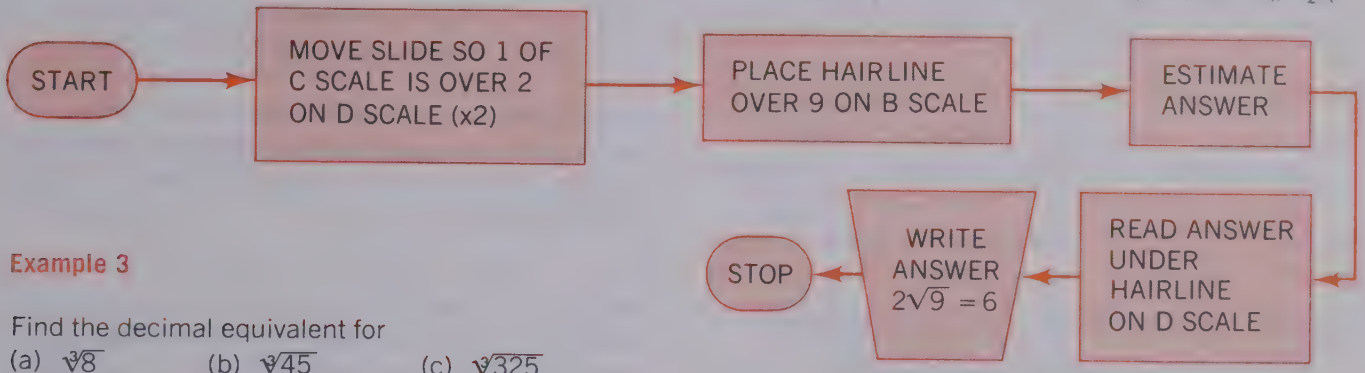
(d) $\sqrt{132}$


START
PLACE HAIR LINE OVER 132 ON SCALE A ON *LEFT* HALF OF SLIDE RULE
ESTIMATE ANSWER
$11^2 = 121$
$12^2 = 144$
$\sqrt{132} \simeq 11$
READ ANSWER ON D SCALE UNDER HAIR-LINE
WRITE ANSWER $\sqrt{132} = 11.5$
STOP

Example 2

Find the decimal equivalent of $2\sqrt{9}$.

Solution


START
MOVE SLIDE SO 1 OF C SCALE IS OVER 2 ON D SCALE (x2)
PLACE HAIRLINE OVER 9 ON B SCALE
ESTIMATE ANSWER
READ ANSWER UNDER HAIRLINE ON D SCALE
WRITE ANSWER $2\sqrt{9} = 6$
STOP

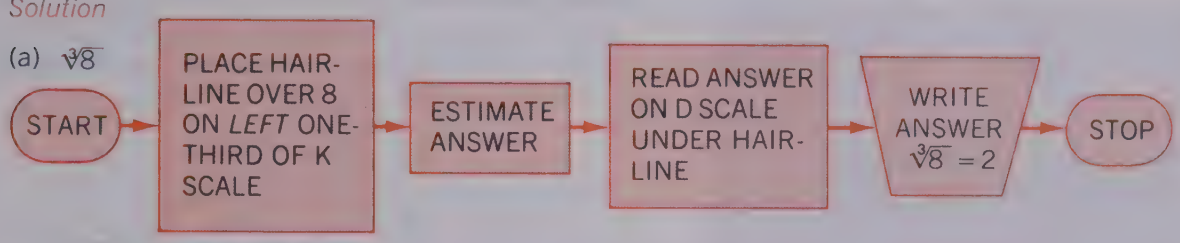
Example 3

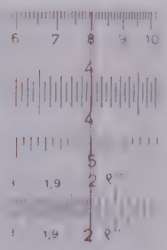
Find the decimal equivalent for

(a) $\sqrt[3]{8}$ (b) $\sqrt[3]{45}$ (c) $\sqrt[3]{325}$

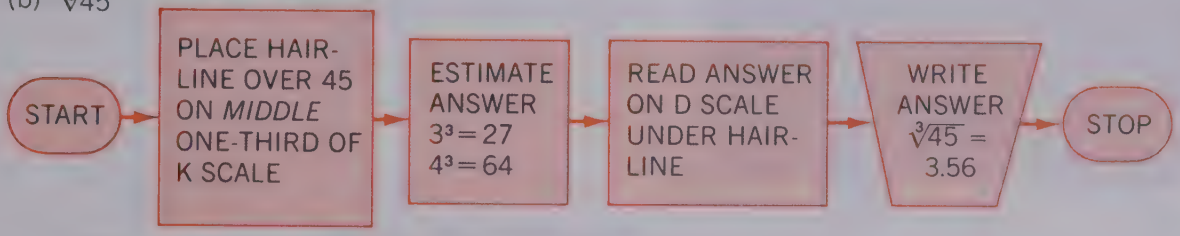
Solution

(a) $\sqrt[3]{8}$

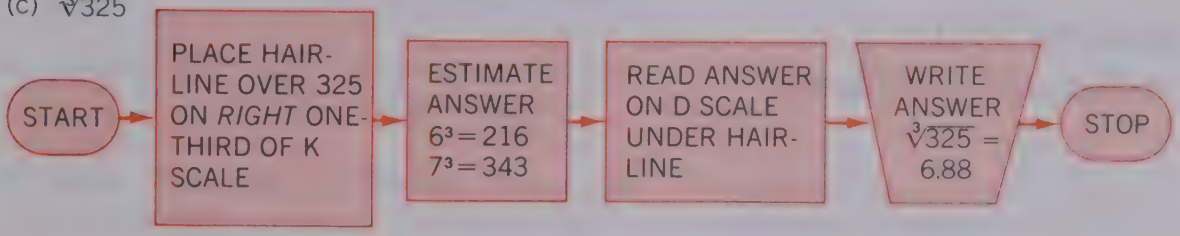

START
PLACE HAIR-LINE OVER 8 ON *LEFT* ONE-THIRD OF K SCALE
ESTIMATE ANSWER
READ ANSWER ON D SCALE UNDER HAIR-LINE
WRITE ANSWER $\sqrt[3]{8} = 2$
STOP

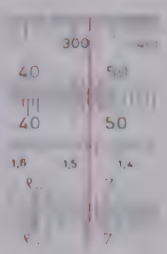


(b) $\sqrt[3]{45}$


START
PLACE HAIR-LINE OVER 45 ON *MIDDLE* ONE-THIRD OF K SCALE
ESTIMATE ANSWER
$3^3 = 27$
$4^3 = 64$
READ ANSWER ON D SCALE UNDER HAIR-LINE
WRITE ANSWER $\sqrt[3]{45} = 3.56$
STOP

(c) $\sqrt[3]{325}$


START
PLACE HAIR-LINE OVER 325 ON *RIGHT* ONE-THIRD OF K SCALE
ESTIMATE ANSWER
$6^3 = 216$
$7^3 = 343$
READ ANSWER ON D SCALE UNDER HAIR-LINE
WRITE ANSWER $\sqrt[3]{325} = 6.88$
STOP



EXERCISE 2·18

Use the slide rule to do the following questions. Find the square root or cube root as indicated.

1. $\sqrt{9}$ 2. $\sqrt{7}$ 3. $\sqrt{450}$ 4. $\sqrt{150}$
5. $\sqrt{635}$ 6. $\sqrt{525}$ 7. $\sqrt{64}$ 8. $\sqrt{85}$
9. $\sqrt{76}$ 10. $\sqrt{33}$ 11. $\sqrt{1246}$ 12. $\sqrt{1600}$
13. $\sqrt{1728}$ 14. $\sqrt{4560}$ 15. $\sqrt{7280}$ 16. $\sqrt[3]{8}$
17. $\sqrt[3]{5}$ 18. $\sqrt[3]{27}$ 19. $\sqrt[3]{81}$ 20. $\sqrt[3]{65}$
21. $\sqrt[3]{45}$ 22. $\sqrt[3]{125}$ 23. $\sqrt[3]{420}$ 24. $\sqrt[3]{650}$
25. $\sqrt[3]{880}$ 26. $\sqrt[3]{1728}$ 27. $\sqrt[3]{8000}$ 28. $\sqrt[3]{27000}$

29. Find the product:
    (a) $2\sqrt{6}$ (b) $3\sqrt{2}$ (c) $5\sqrt{2}$ (d) $8\sqrt{9}$
    (e) $22\sqrt{3}$ (f) $12\sqrt{42}$ (g) $6\sqrt{86}$ (h) $7\sqrt{620}$.

30. Find the product: (Approximate your answers first)
    (a) $35\sqrt{12}$ (b) $42\sqrt{18}$ (c) $144\sqrt{2}$ (d) $356\sqrt{5}$
    (e) $756\sqrt{8}$ (f) $96\sqrt{356}$ (g) $49\sqrt{7434}$.

Project

Project 7

Save and weigh the paper that comes into your home in one week (newspapers, flyers, magazines, etc.). It is estimated there are 5,115,000 households in Canada. If your home is an average, how many pounds of paper is discarded each year in Canada? If the paper is compressed into one-foot cubes weighing 48 pounds, how many cubic feet of waste is there? Suggest the dimensions of a rectangular cuboid that would hold this waste.

2·19 Working With Irrational Numbers

In everyday life, in your science class and in your mathematics class, it is not always convenient nor desirable to work with irrational numbers themselves. It is often more convenient and desirable to use numerals which represent rational approximations of the irrational numbers. For example, if someone wanted a piece of wire 3π inches long or a board $\sqrt{19}$ feet long we might as well use an approximation for 3π and $\sqrt{19}$ – an approximation to the degree of accuracy desired.

We often refer to π as $\frac{22}{7}$. This is a rational approximation for π. Comparing $\frac{22}{7}$ as a decimal with π we have

$$\left.\begin{array}{l}\pi = 3.1415\ldots \text{ or } 3.14 \\ \frac{22}{7} = 3.1428\ldots \text{ or } 3.14\end{array}\right\} \textit{to the nearest hundredth}$$

We note that to the nearest hundredth the approximation is the same as the real value. The value of π to the nearest thousandth is 3.142 and to the nearest ten thousandth is 3.1416.

Similarly $\sqrt{2}$ is 1.4 to the nearest tenth. 1.41 to the nearest hundredth and 1.414 to the nearest thousandth. Any irrational number can be expressed as a decimal fraction to the degree of accuracy desired. Hence, if we wish to approximate $(\sqrt{5} + \sqrt{6})$ to the nearest hundredth we first approximate $\sqrt{5}$ and $\sqrt{6}$ to the nearest thousandth, then add the two and round to the nearest hundredth.

$$\begin{array}{r} \sqrt{5} \simeq 2.236 \\ \sqrt{6} \simeq 2.450 \\ \hline \sqrt{5} + \sqrt{6} \simeq 4.686 \end{array}$$

To the nearest hundredth $\sqrt{5} + \sqrt{6}$ is 4.69

EXERCISE 2·19

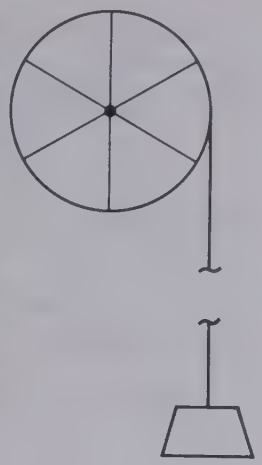
1. Find the approximate value of each to 2 decimal places. (Use tables or slide rule.)
   (a) $\sqrt{5} + \sqrt{6}$ (b) $\sqrt{13} - 2\sqrt{10}$
   (c) $3\sqrt{12} - 3\sqrt{20}$ (d) $4\sqrt{18} + 4\sqrt{24}$
   (e) $3\pi - 4\sqrt{2}$ (f) $\sqrt{3} + \sqrt{7}$

2. A pulley of diameter 3.25 feet has a nylon rope attached to it, as in the diagram. The pulley turns 4 revolutions per minute.

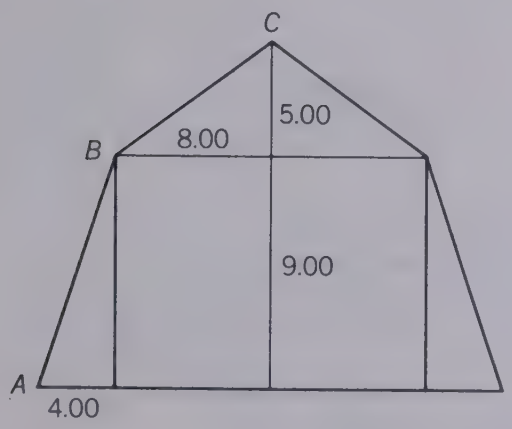
   (a) How many feet does the weight lift in exactly one revolution of the pulley?

   (b) How many feet does the weight lift in one minute? (What is the rate in ft./min. that the weight is being lifted?)

   (c) How many minutes are required to lift the weight 196.00 feet? First give exact answer, then express each answer to the nearest hundredth of a unit.



3. A certain structure being built has a roof shaped as shown below. It is the intention of the builder to place aluminum sheeting on the roof. How many feet is it from *A* to *C* along the roof line?



4. Use a slide rule to calculate the principal square root to 3 digits of:

42	8	39	90	121	150
.36	3.60	36.00	360	3600	

5. Use the slide rule to calculate the cube root of each number in question 4.

6. Use the square root tables in the back of this book to find the principal square root of each of the following:

   3 28 47 55 64 98

7. Use the cube root tables in the back of this book to find the cube root of the numbers in question 6.

8. (a) Find, if possible, the principal square root of each:

   36 47 −2.5 −4 −2.25 56

   (b) Find, if possible, the cube root of each number in (a).

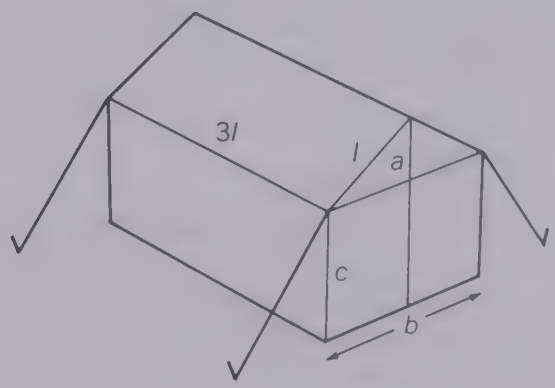
9. Find, if possible, each:

$-\sqrt{49}$	$-\sqrt{144}$	$\sqrt{-25}$	$\sqrt{-144}$
$\sqrt{3^2}$	$\sqrt{296}$	$\sqrt[3]{16}$	$\sqrt[4]{296}$
$\sqrt[3]{125}$	$\sqrt[3]{-125}$	$\sqrt[3]{729}$	$\sqrt[4]{16}$

10. Use Newton's Method with a desk calculator to find the principal square root, to two decimal places, of the following. Then check each answer by multiplication.

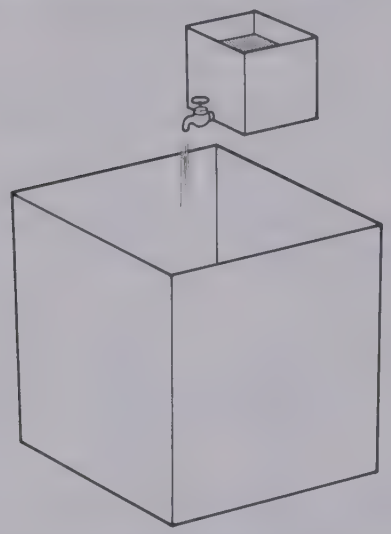
(a) 4300	(b) 38.4	(c) 876.14
(d) 0.056	(e) 0.56	(f) 5.6
(g) 56	(h) 560	(i) 5600

11. A tent is constructed as in the diagram below. The only dimensions given are $a = 4.0$ feet, $b = 8.0$ feet, $c = 3.0$ feet. Find the number of square feet of canvas in the tent. (There are two ends and a floor also.)



12. A circle of diameter 6 inches is compared to one whose diameter is 18 inches. How do the areas of the two circles compare?

13. Water from a filled cubic tank of side 5 feet is transferred to a cubic tank of side 15 feet. What portion of the second tank is filled?



14. A city has a spherical water tank of diameter 60 feet. The city council decided to replace it with spherical tanks of 20 feet in diameter. What is the least number of smaller tanks necessary to hold the amount of water the larger tank holds?
Volume of sphere is given by $\frac{4}{3}\pi r^3$.

15. Express π as a decimal to the nearest
(a) hundred thousandth (b) millionth

Consult the library to obtain the most accurate value for π that you can. Some recent books give the value to fifty decimal places. In 1949 an electronic computer gave the value of π to 2035 places and more recently it has been given to 10,000 places. Do you think the decimal is ever repeated?

16. (a) Devise a mnemonic for the value of π to twelve places.
Here is an example.

Now I seek a group committed to
3 1 4 1 5 9 2
pursue datum, its value ultimate.
6 5 3 5 8

(b) Do the same for $\sqrt{2}$.

2·20 Learning to Use Some Irrational Numbers

Example 1

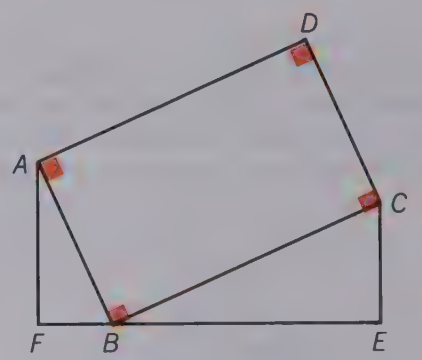
The rectangular city lot *ABCD* is bounded by two right triangles *ABF* and *BEC*. Angles *F* and *E* are right angles. It is known by actual measurement that

$m(\overline{AF}) = 4$ yd. $\quad m(\overline{FB}) = 2$ yd.
$m(\overline{BE}) = 10$ yd. $\quad m(\overline{EC}) = 5.$ yd.

(a) Find exactly the dimensions of the rectangle *ABCD*.

(b) Find the number of lineal yards of fencing necessary to fence the lot *ABCD* (exactly).

(c) Find the area of the rectangular lot *ABCD* (exactly).



Solution

(a) To find the measure of $\overline{BC}$ we use

$$m(\overline{BE})^2 + m(\overline{CE})^2 = m(\overline{BC})^2$$
$$10^2 + 5^2 = m(\overline{BC})^2$$
$$100 + 25 = m(\overline{BC})^2$$
$$m(\overline{BC})^2 = 125$$
$$m(\overline{BC}) = \sqrt{125}$$

It will be left to the student to find $m(\overline{AB})$.
Did you get $\sqrt{20}$?
The dimensions of the lot are $\sqrt{125}$ yards by $\sqrt{20}$ yards.

(b) To find the perimeter:

$$P = 2l + 2w$$

$P = 2m(\overline{BC}) + 2m(\overline{AB})$
$= 2\sqrt{125} + 2\sqrt{20}$
The perimeter is $(2\sqrt{125} + 2\sqrt{20})$ yards.
Do you know how to add these numbers? We shall learn in the next section.

(c) To find the area:
$A = lw$
$= \sqrt{125} \times \sqrt{20}$
The area is $(\sqrt{125} \times \sqrt{20})$ square yards.
Do you know how to multiply these numbers? We shall learn how in this section.

Example 1 illustrated a need for skill in computation with radicals. The following examples illustrate the skills necessary at this time.

Multiplying Radicals

Example 2

Multiply
(a) $\sqrt{4} \times \sqrt{4}$ (b) $\sqrt{25} \times \sqrt{9}$
(c) $\sqrt{6} \times \sqrt{6}$ (d) $\sqrt{7} \times \sqrt{6}$

Solution

Each of (a) and (b) will be done two ways. Compare the two methods.

	Method 1	*Method 2*
(a)	$\sqrt{4} \times \sqrt{4} = 2 \cdot 2 = 4$	$\sqrt{4} \times \sqrt{4} = \sqrt{16} = 4$
(b)	$\sqrt{25} \times \sqrt{9} = 5 \cdot 3 = 15$	$\sqrt{25} \times \sqrt{9} = \sqrt{225} = 15$
(c)	Method 2 is used. Why can we not do this one by Method 1?	$\sqrt{6} \times \sqrt{6} = \sqrt{36} = 6$
(d)		$\sqrt{7} \times \sqrt{6} = \sqrt{42}$

We leave the answer as $\sqrt{42}$ since 42 does not contain a factor which is a perfect square.

From examples (a) and (c) we generalize.

$$\sqrt{n} \times \sqrt{n} = n,\ n \geq 0$$

From examples (b) and (d) we generalize.

$$\sqrt{n} \times \sqrt{m} = \sqrt{nm},\ n, m \geq 0$$

Does the word "irrational" mean the same here as in mathematics? Explain this "irrational".

EXERCISE 2·20

Multiply. Use Method 1.

1. $\sqrt{9} \times \sqrt{9}$
2. $\sqrt{25} \times \sqrt{16}$
3. $\sqrt{36} \times \sqrt{49}$
4. $\sqrt{64} \times \sqrt{25}$
5. $\sqrt{100} \times \sqrt{121}$
6. $\sqrt{49} \times \sqrt{81}$
7. $\sqrt{144} \times \sqrt{225}$
8. $\sqrt{400} \times \sqrt{121}$
9. $\sqrt{4} \times \sqrt{9} \times \sqrt{16}$
10. $\sqrt{81} \times \sqrt{225} \times \sqrt{64}$
11. $\sqrt{25} \times \sqrt{400} \times \sqrt{121}$
12. $\sqrt{64} \times \sqrt{36} \times \sqrt{4} \times \sqrt{25} \times \sqrt{100}$
13. Write in your own words a rule for multiplying radicals of the type in questions 1 through 12.

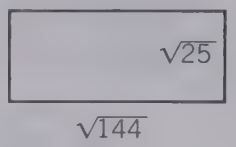
Multiply. Use Method 2.

14. $\sqrt{3} \times \sqrt{7}$
15. $\sqrt{6} \times \sqrt{11}$
16. $\sqrt{15} \times \sqrt{22}$
17. $\sqrt{14} \times \sqrt{25}$
18. $\sqrt{7} \times \sqrt{13}$
19. $\sqrt{3} \times \sqrt{155}$
20. $\sqrt{12} \times \sqrt{5}$
21. $\sqrt{17} \times \sqrt{30}$
22. $\sqrt{3} \times \sqrt{11} \times \sqrt{7}$
23. $\sqrt{13} \times \sqrt{5} \times \sqrt{2}$
24. $\sqrt{10} \times \sqrt{11} \times \sqrt{13}$
25. $\sqrt{2} \times \sqrt{3} \times \sqrt{5} \times \sqrt{7}$

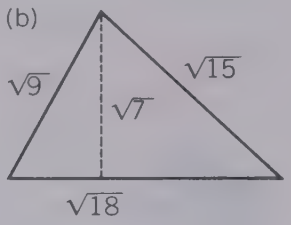
26. Write in your own words a rule for multiplying radicals of the type in questions 14 through 25.

27. Find the area of each figure.

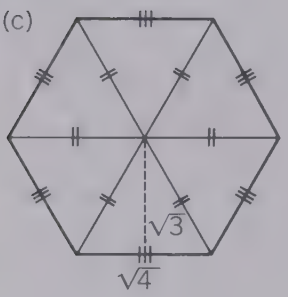
(a)



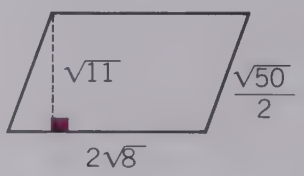
(b)



(c)



(d)



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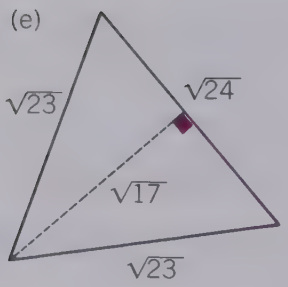
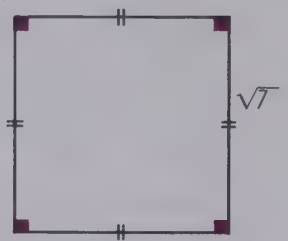


(f)



2·21 Radicals in Simplest Form

The radical $\sqrt{20}$ is not in its simplest form.

$$\sqrt{20} = \sqrt{4 \times 5}$$
$$= \sqrt{4} \times \sqrt{5}$$
$$= 2\sqrt{5}$$

The entire radical $\sqrt{20}$ has been rewritten as a mixed radical $2\sqrt{5}$. Notice the radical was factored so as one of the factors is a perfect square. It is not always apparent what the perfect square factor is. The following statement shows an alternate method to obtain the same answers.

		$\sqrt{20}$
Step 1	Write radicand as product of prime *factors*.	$= \sqrt{2 \times 2 \times 5}$
Step 2	*Associate* common factors in groups of two.	$= (\sqrt{2}\ \sqrt{2}) \times \sqrt{5}$
Step 3	*Multiply* factors that appear twice.	$= 2\sqrt{5}$

Example 1

Write, if possible, each of the following as mixed radicals.

(a) $\sqrt{72}$ (b) $\sqrt{30}$

Solution

(a) 1. Factor. $\sqrt{72} = \sqrt{2 \times 2 \times 2 \times 3 \times 3}$
2. Associate. $= (\sqrt{2}\ \sqrt{2}) \times (\sqrt{3}\ \sqrt{3}) \times \sqrt{2}$
3. Multiply. $= 2 \times 3\sqrt{2}$
4. Multiply. $= 6\sqrt{2}$

(b) 1. Factor. $\sqrt{210} = \sqrt{2 \times 3 \times 5 \times 7}$

No factor appears twice in the factored form of the radicand, therefore the entire radical cannot be rewritten as a mixed radical.

A radical is in *simplest form* when the radicand does not have a factor that is a perfect square.

$$\sqrt{n \times n \times m} = n\sqrt{m}$$

EXERCISE 2·21

Write each as a mixed radical in simplest form.

1. $\sqrt{18}$ 2. $\sqrt{50}$ 3. $\sqrt{75}$ 4. $\sqrt{63}$
5. $\sqrt{108}$ 6 $\sqrt{200}$ 7. $\sqrt{128}$ 8 $\sqrt{288}$
9. $\sqrt{98}$ 10. $\sqrt{242}$ 11. $\sqrt{175}$ 12. $\sqrt{350}$

Write each in simplest form ($a > 0$, $b > 0$, $c > 0$).

13. $\sqrt{8}$ 14. $\sqrt{28}$ 15. $3\sqrt{45}$
16 $5\sqrt{12}$ 17. $\sqrt{22}$ 18 $\sqrt{a^2b}$

19. $\sqrt{a^3b^2}$ 20. $\sqrt{a^4b^3}$ 21. $\sqrt{a^2b^2c^3}$

22. $\sqrt{a^3b^3c^3}$ 23. $\sqrt{4a^3}$ 24. $\sqrt{20b^3c}$

25. Write a flow chart for expressing a radical in simplest form.

26. Express in simplest terms the dimensions of the figures in question 27, Exercise 2·20.

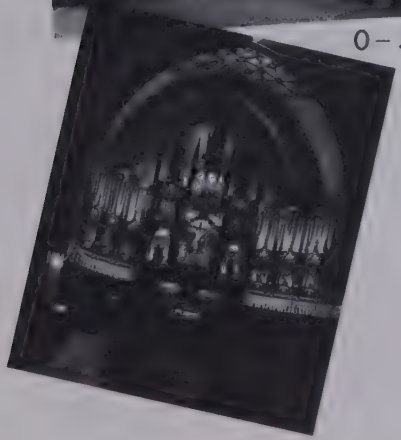
27. Noise pollution is causing man many nervous disorders.

Decibel Ratings of Various Noises

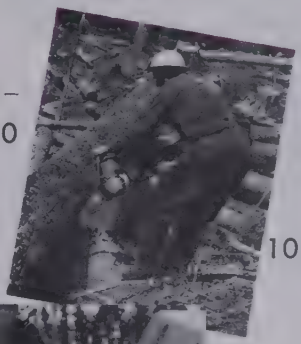
60 – 80

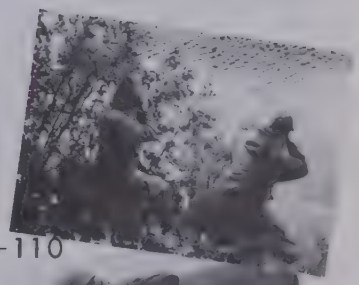
20 – 30

0 – 5



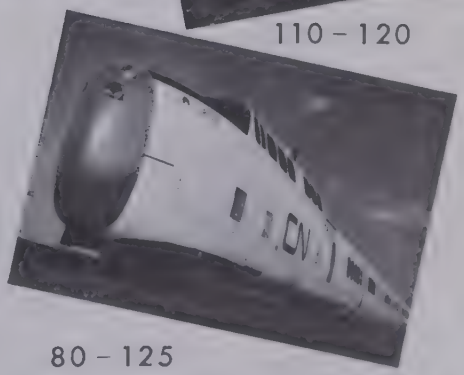
90 – 100





100 – 110

110 – 120



80 – 125

130

A A decibel rating over 110 can cause deafness.

B The pain threshold for humans is 120 decibels.

C Maximum allowable for humans is 130 decibels.

D Constant and continued noises over 80 decibels may cause nervous disorders.

List the number of noises in your community, school and/or home that fall into the four classifications **A**, **B**, **C**, **D**.

What decibel rating do you think a motorbike has when accelerating?

2·22 Addition and Subtraction of Radicals

Like radicals are radicals that have the same index and the same radicand. Expressions with like radicals can be simplified by using the distributive property. Expressions with unlike radicals are left in the form indicating the addition and subtraction.

Example 1

Perform the indicated addition and subtraction where possible:

(a) $3\sqrt{2} + 4\sqrt{2}$

(b) $\sqrt{3} - 2\sqrt{3} + 5\sqrt{3}$

(c) $2\sqrt{2} + 13\sqrt{2} - 3\sqrt{3} + 8\sqrt{2}$.

Solution

(a) $3\sqrt{2} + 4\sqrt{2}$
$= (3 + 4)\sqrt{2}$ *distributive property*
$= 7\sqrt{2}$

(b) $\sqrt{3} - 2\sqrt{3} + 5\sqrt{3}$
$= (1 - 2 + 5)\sqrt{3}$ *distributive property*
$= 4\sqrt{3}$

(c) $2\sqrt{2} + 13\sqrt{3} - 3\sqrt{3} + 8\sqrt{2}$
$= (2\sqrt{2} + 8\sqrt{2}) + (13\sqrt{3} - 3\sqrt{3})$ *associative, commutative*
$= (2 + 8)\sqrt{2} + (13 - 3)\sqrt{3}$ *distributive*
$= 10\sqrt{2} + 10\sqrt{3}$
$= 10(\sqrt{2} + \sqrt{3})$ *distributive*

Since $\sqrt{2}$ and $\sqrt{3}$ are unlike radicals, $10(\sqrt{2} + \sqrt{3})$ cannot be simplified further.

It is not always obvious whether radicals are like or unlike. One should always write each radical in simplest form before attempting to add or subtract.

Example 2

Perform the indicated addition or subtraction where possible.

(a) $\sqrt{18} - \sqrt{50}$

(b) $\sqrt{20} - \sqrt{12} + \sqrt{45} - \sqrt{75}$

Solution

(a) $\sqrt{18} - \sqrt{50}$
$= \sqrt{3 \times 3 \times 2} - \sqrt{2 \times 5 \times 5}$
$= (\sqrt{3}\,\sqrt{3})\sqrt{2} - (\sqrt{5}\,\sqrt{5})\sqrt{2}$
$= 3\sqrt{2} - 5\sqrt{2}$
$= (3 - 5)\sqrt{2}$
$= -2\sqrt{2}$

(b) $\sqrt{20} - \sqrt{12} + \sqrt{45} - \sqrt{75}$
$= \sqrt{2 \times 2 \times 5} + \sqrt{3 \times 3 \times 5} - \sqrt{2 \times 2 \times 3} - \sqrt{5 \times 5 \times 3}$
$= (\sqrt{2}\,\sqrt{2})\sqrt{5} - (\sqrt{2}\,\sqrt{2})\sqrt{3} + (\sqrt{3}\,\sqrt{3})\sqrt{5} - (\sqrt{5}\,\sqrt{5})\sqrt{3}$
$= 2\sqrt{5} - 2\sqrt{3} + 3\sqrt{5} - 5\sqrt{3}$
$= 2\sqrt{5} + 3\sqrt{5} - 2\sqrt{3} - 5\sqrt{3}$
$= (2 + 3)\sqrt{5} - (2 + 5)\sqrt{3}$
$= 5\sqrt{5} - 7\sqrt{3}$

Since $\sqrt{5}$ and $\sqrt{3}$ are unlike radicals this problem cannot be simplified further.

EXERCISE 2·22

Perform the indicated addition and subtraction where possible.

1. $2\sqrt{3} + 35\sqrt{3}$
2. $24\sqrt{11} - 18\sqrt{11}$
3. $3\sqrt{5} - 2\sqrt{5} + 3\sqrt{5}$
4. $13\sqrt{7} - 2\sqrt{13} - 2\sqrt{7} + 11\sqrt{3}$
5. $5\sqrt{13} - 6\sqrt{17} - 8\sqrt{17} + 9\sqrt{13}$
6. $2\sqrt{3} + 3\sqrt{5} - 4\sqrt{6} - 5\sqrt{14}$
7. $3\sqrt{ab} - 2\sqrt{ab} + 5\sqrt{ab} - 6\sqrt{ab}$
8. $\sqrt{51} - \sqrt{41} + 2\sqrt{41} - 13\sqrt{51}$
9. $\sqrt{12} - \sqrt{48}$
10. $2\sqrt{98} + 3\sqrt{12} - 4\sqrt{108}$
11. $\sqrt{80} - 2\sqrt{45} + 3\sqrt{20}$
12. $2\sqrt{12} - 3\sqrt{3} - \sqrt{75}$

13. $5\sqrt{52} - 4\sqrt{13}$

14. $3\sqrt{64} - 2\sqrt{2} - 3\sqrt{8}$

15. $\sqrt{180} - 3\sqrt{80} - 2\sqrt{20} + 3\sqrt{45}$

16. $\sqrt{98} + 2\sqrt{108} + 2\sqrt{12} - \sqrt{50} - 6\sqrt{3}$

17. Find the perimeter in simplest form of each figure in question 27 of Exercise 2·20.

2·23 More Multiplication (Optional)

In Section 2·20 you learned to do some simple multiplication with radicals. We will now use information gained in Sections 2·20 and 2·21 to simplify multiplication in certain cases.

Example 1

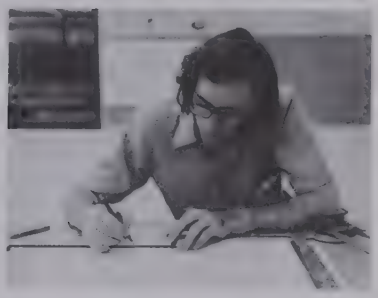
Find the product in each instance.
(a) $2\sqrt{3} \times 4\sqrt{5}$
(b) $\sqrt{20} \times \sqrt{32}$
(c) $(\sqrt{2} - \sqrt{3})(\sqrt{5} + \sqrt{2})$

Solution

(a) $2\sqrt{3} \times 4\sqrt{5}$
$= 2 \times \sqrt{3} \times 4 \times \sqrt{5}$
$= 2 \times 4 \times \sqrt{3} \times \sqrt{5}$ {commutative, associative.
$= 8 \times \sqrt{15}$
$= 8\sqrt{15}$

(b) $\sqrt{20} \times \sqrt{32}$
$= 2\sqrt{5} \times 4\sqrt{2}$ *Why?*
$= 2 \times \sqrt{5} \times 4 \times \sqrt{2}$
$= 2 \times 4 \times \sqrt{5} \times \sqrt{2}$ *Why?*
$= 8 \times \sqrt{10}$
$= 8\sqrt{10}$

(c) $(\sqrt{2} + \sqrt{3})(\sqrt{2} + \sqrt{3})$
$= \sqrt{2}(\sqrt{2} + \sqrt{3}) + \sqrt{3}(\sqrt{2} + \sqrt{3})$ *dist.*
$= \sqrt{2}\sqrt{2} + \sqrt{2}\sqrt{3} + \sqrt{3}\sqrt{2} + \sqrt{3}\sqrt{3}$ *dist.*
$= 2 + \sqrt{6} + \sqrt{6} + 3$
$= 2 + 3 + \sqrt{6} + \sqrt{6}$ {commutative, associative
$= 5 + 2\sqrt{6}$



Why would a knowledge of irrational numbers be helpful or essential to these people?

EXERCISE 2·23

Read at sight the product in each of the following.

1. $(3\sqrt{2})(4\sqrt{3})$
2. $(3\sqrt{6})(5\sqrt{2})$
3. $(4\sqrt{5})(9\sqrt{7})$
4. $(a\sqrt{2})(b\sqrt{5})$
5. $(x\sqrt{7})(y\sqrt{11})$
6. $(2\sqrt{x})(9\sqrt{y})$
7. $(3\sqrt{5})^2$
8. $(2\sqrt{3})(4\sqrt{5})(7\sqrt{11})$
9. $(a\sqrt{x})(b\sqrt{w})$
10. $(m\sqrt{x})(m\sqrt{x})$

Express the product in simplest form.

11. $(\sqrt{12})(\sqrt{108})$
12. $(\sqrt{500})(\sqrt{180})$
13. $(\sqrt{50})(\sqrt{98})$
14. $(\sqrt{4a^2b})(\sqrt{18a^2})$
15. $(\sqrt{16ab^2})(\sqrt{32ab^2})$
16. $(\sqrt{9a})(\sqrt{81a})(\sqrt{27a^2})$
17. $\sqrt{5}(\sqrt{3} - 1)$
18. $\sqrt{7}(\sqrt{7} + 3)$
19. $\sqrt{6}(\sqrt{2} + \sqrt{3} - \sqrt{12})$
20. $\sqrt{x}(\sqrt{y} - \sqrt{x} + \sqrt{w})$
21. $(\sqrt{x} + \sqrt{y})^2$
22. $(\sqrt{20} - \sqrt{18})^2$
23. $5\sqrt{3}(\sqrt{2} - 2\sqrt{5})$
24. $(\sqrt{3} - 2\sqrt{2})(2\sqrt{3} - 3\sqrt{2})$

25. $(\sqrt{5} - \sqrt{2})(\sqrt{2}\sqrt{5} + 3\sqrt{2})$

26. $(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3})$

27. $(\sqrt{5} - \sqrt{13})(\sqrt{5} + \sqrt{13})$

28. $(2\sqrt{2} - 3\sqrt{3})(2\sqrt{2} + 3\sqrt{3})$

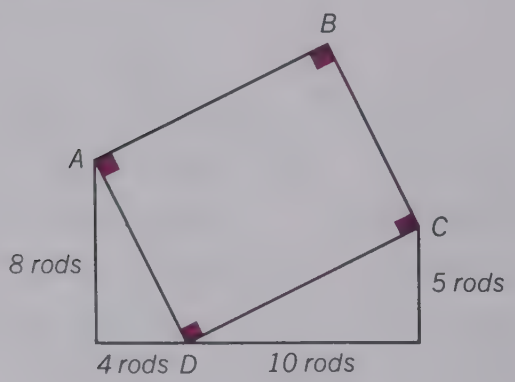
29. $(5\sqrt{7} - 4\sqrt{10})(5\sqrt{7} + 4\sqrt{10})$

30. Find the area of a square whose side is $(3 + 4\sqrt{5})$ inches.

31. Find the area of a rectangle whose length and width are $\sqrt{18}$ and $(\sqrt{3} + \sqrt{5})$ inches.

32. By squaring $\sqrt{7} + \sqrt{3}$ and $\sqrt{4} + \sqrt{6}$, find which is the greater.

33. Find the perimeter and the area of the rectangular city lot *ABCD*.



2·24 Evaluating Irrationals

In practical applications it is often convenient to use numerals which represent rational approximations of irrational numbers.

For example, in Example 1 in Section 2·20, an engineer would not likely use $\sqrt{125}$ and $\sqrt{20}$ as the length and width of the lot. He would more likely use the approximation

$$\sqrt{5} = 2.236$$

Then, noting that $\sqrt{125} = 5\sqrt{5}$ and $\sqrt{20} = 2\sqrt{5}$ he would then write

$$\begin{aligned} 5\sqrt{5} &\simeq 5(2.236) \\ &= 11.210 \\ 2\sqrt{5} &\simeq 2(2.236) \\ &= 4.472 \end{aligned}$$

Example 1

Given $\sqrt{2} \simeq 1.414$ and $\sqrt{3} \simeq 1.732$, evaluate
(a) $\sqrt{18}$ (b) $\sqrt{6}$

Solution

(a) $\sqrt{18} = \sqrt{3 \times 3 \times 2}$
$= 3\sqrt{2}$
$\sqrt{18} \simeq 3(1.414)$
$\therefore \sqrt{18} \simeq 4.242$

(b) $\sqrt{6} = \sqrt{3} \times \sqrt{2}$
$\sqrt{6} \simeq (1.732)(1.414)$
$\therefore \sqrt{6} \simeq 2.449$

EXERCISE 2·24

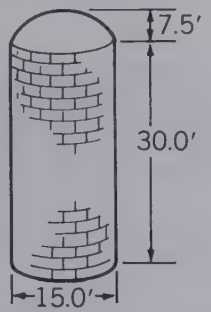
Given $\sqrt{2} \simeq 1.414$, $\sqrt{3} \simeq 1.732$, $\sqrt{5} \simeq 2.236$, find the approximate value of each of the following.

1. $\sqrt{6}$
2. $\sqrt{8}$
3. $\sqrt{10}$
4. $\sqrt{15}$
5. $\dfrac{\sqrt{9}}{\sqrt{5}}$
6. $\sqrt{75}$
7. $\sqrt{60}$
8. $\sqrt{30}$
9. $\sqrt{24}$
10. $\sqrt{27}$
11. $(\sqrt{2} + \sqrt{3})$
12. $\sqrt{8} + \sqrt{6}$
13. $2\sqrt{15} + 3\sqrt{30}$
14. $\sqrt{75} - \sqrt{27}$
15. $\sqrt{8} - \sqrt{27}$
16. $\sqrt{80} - \sqrt{10}$
17. $\sqrt{6} \times \sqrt{18}$
18. $\sqrt{10} \times \sqrt{15}$
19. $\sqrt{27} \times \sqrt{125}$

20. Refer to question 33, Exercise 2·23. Find the decimal approximations for
(a) the length and width of the rectangular lot,
(b) the area of the lot,
(c) the perimeter of the lot.

21. A natural gas company wishes to replace a cylindrical tank, diameter of 10.0 feet and height of 15.0 feet, with a spherical tank of the same volume. What must the diameter of the spherical tank be?

22. A farmer wishes to paint the silo shown in the diagram. The top is a hemisphere. Each gallon of paint covers 600 square feet. How many gallons of paint are needed? (Area of sphere is given by $4\pi r^2$.)



23. An oil refinery has 8 spherical storage tanks each with a diameter of 40 feet. The company decides to double its storage capacity by building one large spherical tank. What must the diameter of this larger tank be?

Using the Library

1. Read *Wonderful World of Mathematics*, pages 22 and 23. Make a simple abacus and learn how to make simple computations on it. Also see question 2.
2. Read *Life Science Library Mathematics*, pages 18, 19, 20, and 21. This explains the abacus in more detail.
   Read page 25. Write some numbers in the binary system. Explain each.
   Read page 30. Explain how the computer divides 50 by 7.

REVIEW EXERCISE 2

1. What is the solution set of the open sentence $x^2 = 2$?

2. Write a fraction numeral that names the same number as
   (a) $\frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{2}}{\sqrt{3}}$ (b) $\sqrt{\frac{4}{9}}$.

3. Simplify:
   (a) $\sqrt{384}$ (b) $3\sqrt{98} - 7\sqrt{18}$ (c) $\sqrt[4]{9}$.

4. Given $\sqrt{5} \simeq 2.24$ find an approximation for $\sqrt{500}$.

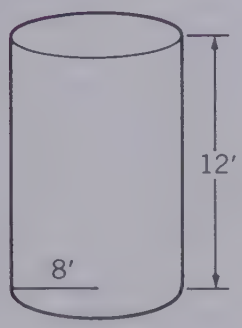
5. Using Newton's Method or slide rule find an approximation to 3 digits of $\sqrt{34}$.

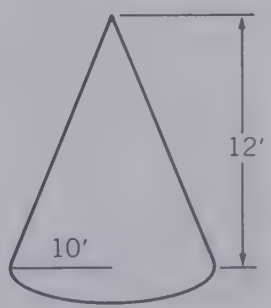
6. By geometric construction find a unique point in the number line that corresponds to $\sqrt{6}$.

7. How is the phrase "$-\sqrt{9} = -3$" read?

8. The surface area of a sphere is $4\pi r^2$. What is the surface area of a sphere with radius 14.20 inches? ($\pi = 3.14$)

9. Two tanks, one shaped like a cylinder and one like a cone, are filled with jet fuel. The depth of both tanks is 12.0 feet. The radius of the cylindrical tank is 8.0 feet and that of the conical tank is 10.0 feet. How much more fuel is in the larger tank than in the smaller? Calculate approximate value to 3 digits. (Volume of cone is given by $V = \frac{1}{3}\pi r^2 h$; volume of cylinder is given by $V = \pi r^2 h$. Use $\pi = 3.14$ unless slide ruler is used.)





10. In 1968 there were 862,000 vehicles (not including motorcycles) reported involved in accidents. There were 5,071 deaths and 165,825 injuries. In the same year there were 8,052 motorcycles reported involved in accidents. There were 247 motorcyclist deaths and 8,076 injuries.

Calculate the $\frac{\text{death}}{\text{accident}}$ and $\frac{\text{injuries}}{\text{accidents}}$ ratios expressed in decimal form to one decimal for motorcycles and for all other vehicles.

What is your conclusion about safety on motorcycles compared to safety in other vehicles?

Chapter Three
RELATIONS

3·1 Introduction

A hockey enthusiast, in reading the sports page, found the final standings in the Eastern Division of the NHL listed:

Montreal	94
New York	90
Boston	84
Toronto	80
Vancouver	76
Detroit	66
Buffalo	60

Being a student of mathematics, he noted that this information could have been listed as a set of pairs:

{(Montreal, 94), (New York, 90),
(Boston, 84), (Toronto, 80), (Vancouver, 76),
(Detroit, 66), (Buffalo, 60)}

The first member of each pair is the name of the club while the second member of each pair is the total points earned. The pair (Montreal, 94) tells us that Montreal *earned* 94 points; (New York, 90) tells us that New York *earned* 90 points; and so on. Notice the implied word that connects the two members of the pair is *earned*. We must say "Montreal *earned* 94 points" and not "94 points earned Montreal". If the words connecting the pair were *was earned by* then the pair would be written (94, Montreal). Therefore, the word connecting the members of the pair being understood, the order of the pair is important. Since the order of the members in the pair is important we shall call sets of this kind sets of ordered pairs.

What do you think of when you hear the word relation? No doubt you think of members of your family: your brothers, sisters, parents, cousins, uncles, and so on. The word may bring to mind the ways in which one individual is related to another. For example:

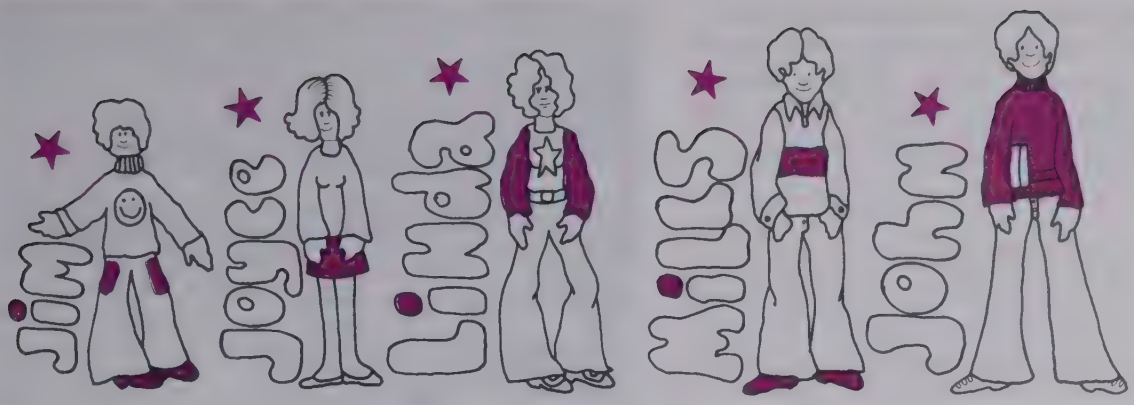
Mary *is the cousin of* Mark.
Kathy *is the older sister of* John.
Marion *is taller than* Jose.
Jack *likes* Jill.

You may think of the way in which a person or object is related to some other object:

Mark *owns* the red Chevy.
Mike *is the captain of* the basketball team.
Four pounds of candy *cost* $4.80.
Winnipeg *has an average annual precipitation of* 27 inches.

The underlined portion of each statement expresses a relation. There are many words or phrases which show relations. Each underlined portion above indicates a relation between two things or groups of things. For this reason we refer to these relations as *binary relations*. A binary relation links a pair of things. If we speak of the relation between three things we have a ternary relation. We will illustrate this in an example later. In the same manner we may speak of relations involving more than three things.

Let us consider a particular relation, *is taller than*, as it applies to a group of five students in a class. If we arranged the students from tallest to shortest as in the diagram it would be easy to see who is the tallest. We could then list all the pairs of students such that the student first named in any pair is taller than the student named second. We would have (John, Mills), (John, Linda), (John, Joyce), (John, Jim), (Mills, Linda), (Mills, Joyce), (Mills, Jim), (Linda, Joyce), (Linda, Jim), (Joyce, Jim). Since the first person named must be taller than the second person named, the order of the names is very important.



For this reason we call each pair an *ordered pair*. The set of first components of ordered pairs in a relation is the DOMAIN of the relation.

The set of second components of ordered pairs in a relation is the RANGE of the relation.

Example 1

Tabulate the domain and the range.

$\{(25,-5), (25,5), (16,4), (121,11), (9,3), (9,-3)\}$

Solution

The domain is $\{25, 16, 121, 9\}$.
The range is $\{-5, 5, 4, 11, -3, 3\}$.

You will recall that an element need only be listed once in a set – hence the reason 25 and 9 are each listed once only in the domain set.

Example 2

Find the domain and range for the relation

$$R = \{(x, y) \mid y = 2x, x \in N\}$$

Solution

You will recall that this relation says the set of all ordered pairs (x,y) such that y is equal to twice x and the replacement for x is the set of natural numbers. Hence the domain is the set of natural numbers. Some of the ordered pairs are:

$$(1,2), (2,4), (3,6), (4,8), (5,10).$$

From this one can see that the range is the set of even natural numbers.

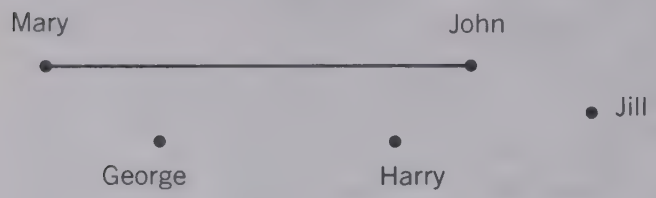
EXERCISE 3·1

1. State the domain and range of each:
   (a) $\{(2,3), (3,4), (7,8), (11,13)\}$
   (b) $\{(1,3), (3,5), (5,7), (7,9), \ldots\}$
   (c) $\{(2,6), (4,8), (6,10), (8,12), \ldots\}$
   (d) $\{(0,0), (1,1), (2,4), (4,16), \ldots\}$

2. State the domain and range of each:
   (a) $\{(x,y) \mid y = x^3,\ x \in I^+,\ x < 9\}$
   (b) $\{(x,y) \mid y = x^3,\ x \in I\}$
   (c) $\{(v,w) \mid w = 3v + 2,\ v \in R\}$

3. In each of the following identify the sets connected by the relation by underlining once the words indicating the sets and twice the words indicating the relation.
   (a) Mike drinks Coca-Cola.
   (b) Harry lives next door to Johnny Bower.
   (c) The Toronto Maple Leafs is the best team in the NHL.
   (d) The trees are taller than the fence.

4. Consider the family of 5 children who are from the oldest to the youngest: Mary, John, Jill, Harry, George. Represent each person by a dot as shown. Draw lines to represent the relation *R, is a brother of.* The line joining Mary and John is drawn.



(a) Is Mary a brother of John?
(b) Is John a brother of Mary?
(c) How might we indicate on the line that John is a brother of Mary so as it will not indicate Mary is a brother of John?
(d) Complete the diagram.
(e) List by means of ordered pairs the relation *R – is a brother of.*
(f) List the first elements of the ordered pairs of the relation *R*.
(g) List the second elements of the ordered pairs for the following:
(i) R_1: *is the sister of*
(ii) R_2: *is the younger sister of*

5. Study the scale drawings of the following cars. Then list the ordered pairs to express the relation, *is longer than.*

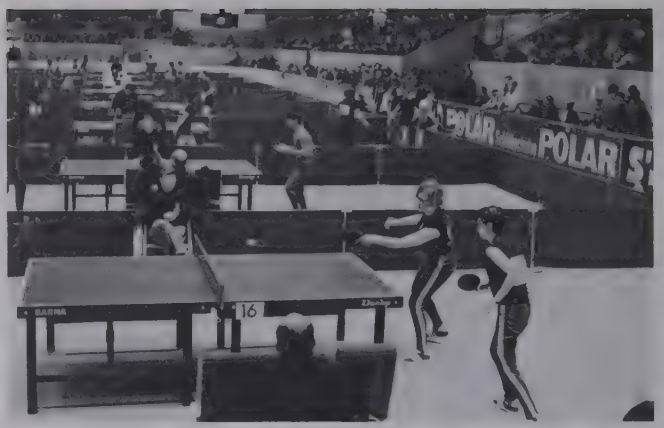
Vega

Mercury

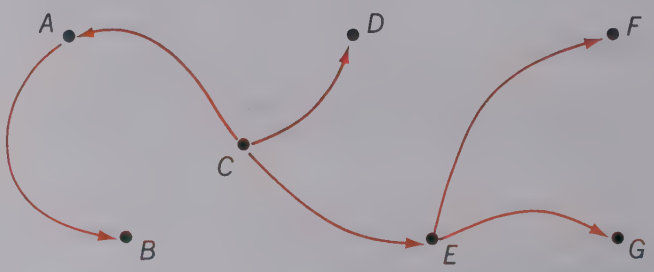
Mercedes

Volkswagen

Cadillac



6. The following figure represents the relation *beat* in a set of 7 table tennis players in a knockout tournament. Who won and who was the runner-up? Make the first component the winner and the second component the loser so as to express the relation *beat.*



7. In order, the youngest first, Mike, Henry, Dolores, and Jake are the four children in a family. Represent on a diagram the relation, *is younger than.* How many ordered pairs are in the relation, *is younger than*?

8. Obtain the height of each student in your row. Construct a dot-arrow graph as in question 4, to represent the relation *is shorter than.* List the ordered pairs in the relation. Compare your answers with those that other students in your row have.

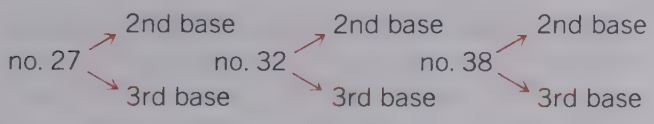
9. Given the set $M = \{2, -3, 5, -1, 0, 3\}$ list the ordered pairs of the relation
(a) R_1: *is greater than.*
(b) R_2: *is less than.*
(c) R_3: *is the square of.*
(d) R_4: *is the additive inverse of.*
(e) R_5: *is the next larger consecutive integer to.*

3·2 Cartesian Product

In choosing the intercity baseball team, the coach finds he has the line-up complete except for second base and third base. In the dugout sit Roger, number 27; Dereck, number 32; and Miller, number 38. He contemplates the ways he can complete the line-up. He thinks:

number 27 – either 2nd base or 3rd base
or number 32 – either 2nd base or 3rd base
or number 38 – either 2nd base or 3rd base.

The coach has matched each player with each position. Another way to represent this is by this diagram:



Let us now present this problem the way a mathematician would. First, list the players by the set

$$B = \{27, 32, 38\}$$

and the positions to be played by the set

$$P = \{2\text{nd}, 3\text{rd}\}$$

Next relate these two sets:

b plays *p*, where *b* is a replacement from set *B* and *p* is a replacement from set *P*. (You will recall that sets are designated by capital letters and members of a set by small letters.) The relation can be expressed by the ordered pair (*b*,*p*). Hence, the set of ordered pairs is:

{(27,2nd), (27,3rd), (32,2nd),
(32,3rd), (38,2nd), (38,3rd)}

This set is called the *cross product* of the set *B* with the set *P*. It is written

$B \times P$ = {(27,2nd), (27,3rd), (32,2nd),
(32,3rd), (38,2nd), (38,3rd)}

The symbols $B \times P$ is read "*B* cross *P*". The set $B \times P$ is sometimes called the *Cartesian product* of sets *B* and *P*.

The *Cartesian product*, $B \times P$, is the set of all ordered pairs in which every element of *B* is matched with every element of *P*. The set of elements in *B* is the domain and the set of elements in *P*, the range.

Example 1

Given the two sets:

$A = \{8, 27, 64\}$ $B = \{2, 3, 4\}$

(a) show by diagram the Cartesian product $A \times B$;
(b) tabulate $A \times B$;
(c) tabulate $B \times A$.
(d) Which members of the set $A \times B$ are used to obtain true statements for "*a* is the cube of *b*"?
(e) Which members of $B \times A$ are used to obtain true statements for "*b* is the cube root of *a*"?

Solution

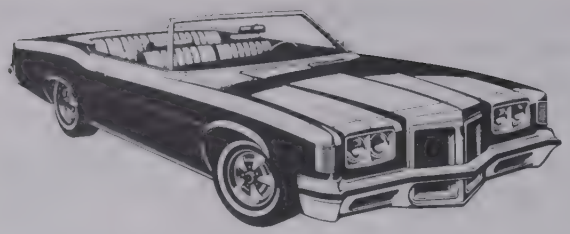
(a) $A \times B$ $8 \to 2, 3, 4$ $27 \to 2, 3, 4$ $64 \to 2, 3, 4$

(b) $A \times B$ = {(8,2), (8,3), (8,4), (27,2),
(27,3), (27,4), (64,2), (64,3), (64,4)}

(c) $B \times A$ = {(2,8), (3,8), (4,8), (2,27),
(3,27), (4,27), (2,64), (3,64), (4,64)}

(d) The ordered pairs (8,2), (27,3), (64,4) produce true statements for "*a* is the cube of *b*."

(e) The ordered pairs (2,8), (3,27), (4,64) produce true statements for "*b* is the cube root of *a*."



Cars are manufactured in many basic styles with a variety of exterior colors, interior colors and extra accessories. A person may order a convertible, metallic green exterior, matching green interior and with a radio, power steering and power brakes. Let us simplify the situation and express it mathematically.

Example 2

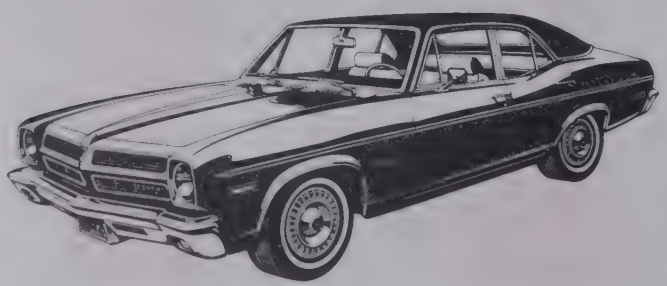
Represent the set of basic styles by S:
$S = \{\text{sedan, convertible, sports}\}$.
Represent the set of exterior colors by C;
$C = \{\text{Royal blue, Aztec bronze, black}\}$.

(a) Using only these two sets, tabulate the various kinds of cars available.
(b) How many kinds are available?

Solution

(a) We wish to match each basic style with each color.
This is the same as $S \times C$.
$S \times C = \{$(sedan, Royal Blue), (sedan, Aztec bronze), (sedan, black), (convertible, Royal blue), (convertible, Aztec bronze), (convertible, black), (sports, Royal blue), (sports, Aztec bronze), (sports, black)$\}$.

(b) There are 9 different kinds.
$S \times C = 3 \times 3 = 9$ pairs.



Example 3

List in a set the ordered pairs in
$\{(g,h) | g,h \in K\}$ where $K = \{2, 3, 4\}$.

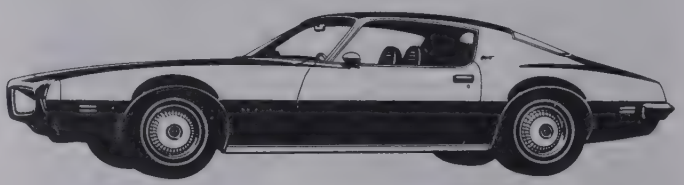
Solution

In this question we must match each g with each h. Since g and h are to be replaced with the elements from K, the question asks us to find $K \times K$.

$K \times K = \{(2,2), (2,3), (2,4), (3,2), (3,3), (3,4), (4,2), (4,3), (4,4)\}$.

EXERCISE 3·2

1. A car manufacturer produces cars at the request of the buyers. The customer chooses one component from each of the sets. How many kinds of cars can the manufacturer produce? Show by diagrams all the different models. Then represent, by ordered triples, the set $S \times C \times I$.



style: $S = \{\text{sports, sedan}\}$
exterior color: $C = \{\text{white, emerald, onyx}\}$
interior color: $I = \{\text{white, forest}\}$

2. In the United Fund drive the accumulated collections at the end of each week were as follows:

First week – \$9,000 Second week – \$20,000
Third week – \$25,000 Fourth week – \$30,000

   (a) Write as ordered pairs the above information considering the weeks to be the domain.
   (b) Write as ordered pairs the above information considering the collections to be the domain.

3. Complete the table below so that it indicates the relation "a is the largest prime in b."

a	7		3	13				
b	14	88	6	39	51	25	47	56

4. The domain of a relation is M and the range is N.
$M = \{\alpha, \beta, r\}$ $N = \{\pi, \tau, \phi, \theta, \ni\}$
How many members in the set $M \times N$?

5. Given the sets M and N. Find the Cartesian product $M \times N$.
Tabulate $N \times M$.
$M = \{\blacktriangle, \blacktriangledown, \blacksquare, \blacklozenge\}$
$N = \{\bullet, \text{⬬}, \text{⬮}\}$

6. Tabulate the cross product $A \times B$ given
$A = \{x,y,z\}$ and $B = \{1,2\}$.
(a) How many pairs are there in $A \times B$?
(b) In how many pairs does x occur?

7. List the Cartesian product $A \times A$, given
$A = \{5,6,7\}$.
How many elements are there in $A \times A$?

8. Given $M = \{19, 20, 21, 22\}$ and $N = \{a, b, c\}$, how many elements are there in
(a) $M \times M$ (b) $M \times N$ (c) $N \times M$ (d) $N \times N$?

9. If there are 7 elements in W, 5 in Y and 4 in Z, how many ordered pairs are there in
$W \times Y$? $Y \times Y$? $Y \times Z$? $W \times Z$?

10. $H = \{1, 2, 3, 5, 6, 10, 11, 12\}$.
Tabulate in set form, all the ordered pairs such that the first member is 2 larger than the second member.

11. A merchant sells golf balls at \$1.35 each. He wishes to post a price list showing the cost of 1, 2, 3 and so on to 10 balls. Construct a table showing this information.

number (n)	1	2	3	. . .	10
cost (c)					

Do the ordered pairs so formed satisfy the relation
$\{(x,c) | c = 1.35x,\ x \in N,\ x \leq 10\}$?

12. Tabulate the Cartesian product $A \times B$ where
$A = \{5, 3, 7, 1\}$ and $B = \{4, 2, 6\}$.
Which members of the Cartesian product may be used to obtain true statements for
(a) a is less than b?
(b) a is greater than b?
(c) a is one more than the square of b?
(d) a is three times larger than b?
(e) a is 7 less than twice b?

13. Refer to the chart of decibel ratings on page 55 in Section 2-21. Show the information by means of a bar graph.

Tabulate 5 elements in the set **P**, *when each element is a type of pollution. Tabulate 5 elements in the set* **C** *when each element is a natural resource being harmed. Which elements of* **P** × **C** *indicate the relation "harms" as in* **P** *harms* **C**.

3·3 Mappings

The following examples illustrate relations as mappings.

Example 1

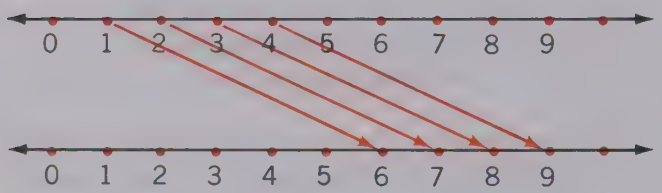
Given the domain $K = \{0, 1, 2, 3\}$ and the relation "k add 5 is h." Show this relation by a mapping diagram. Tabulate the relation as a set of ordered pairs.

Solution

Making replacements for k from the set K we get

0 add 5 is 5; hence $0 \rightarrow 5$
1 add 5 is 6; hence $1 \rightarrow 6$
2 add 5 is 7; hence $2 \rightarrow 7$
3 add 5 is 8; hence $3 \rightarrow 8$

A mapping diagram is shown below on two number lines:



Notice that in $2 \rightarrow 7$, 2 is in the domain – the tail of the arrow is at 2. The head of the arrow is at 7. $2 \rightarrow 7$ is read "2 maps onto 7."

The set of ordered pairs is

$$\{(0, 5), (1, 6), (2, 7), (3, 8)\}$$

Example 2

In the Cooksville High Students' Union the election results were:

George as president, Maybelle as secretary, Hank as treasurer, Margaret on the yearbook committee, Josie on the yearbook committee.

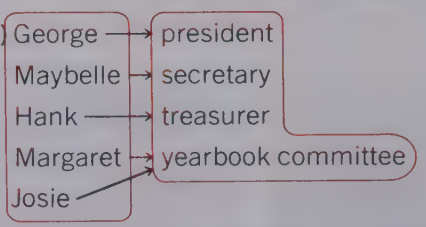
Show the relation between the students and the positions in the students' union by

(a) arrow notation (b) diagram.

Solution

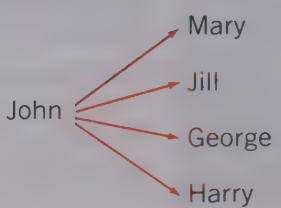
(a) George → president
Maybelle → secretary
Hank → treasurer
Margaret → yearbook committee
Josie → yearbook committee

(b)

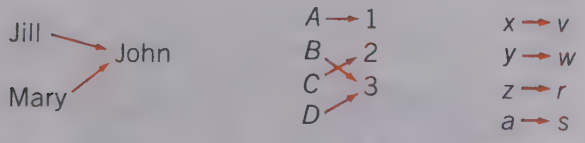


Since George is president, the arrow links *George* and *president*. Margaret and Josie are both linked with the yearbook committee.

A mapping is a special kind of relation. Earlier in the section on relations we showed this relation *is a brother of:*



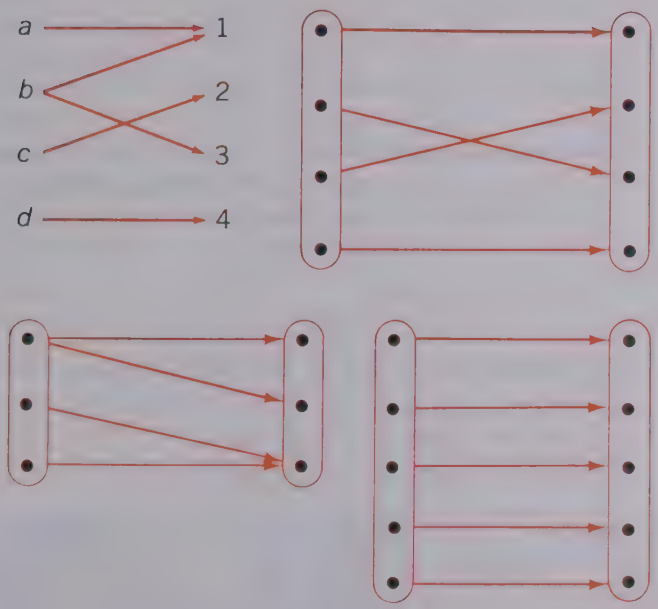
Here we had more than one arrow going from John. In a mapping there is only one arrow going *from* each member in the domain. More than one arrow may go *to* an element in the range. The relations below are mappings.



The symbol $x \rightarrow x^2 + 2$ indicates the mappings of the members from the set X into the set identified by $x^2 + 2$. The first member of the ordered pair is x and the second member is $x^2 + 2$. If $X = \{2, 3, 4\}$ then the set of ordered pairs satisfying the mapping $x \rightarrow x^2 + 2$ is $\{(2, 6), (3, 11), (4, 18)\}$.
The ordered pair in general is $(x, x^2 + 2)$.

EXERCISE 3·3

1. Which of the following are mappings?



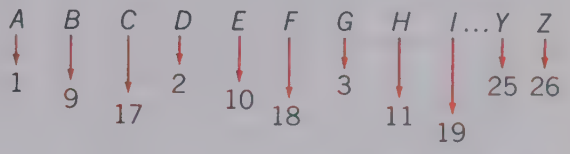
2. An operation maps the number y into the number $3y - 2$.
   (a) Show the mappings when $Y = \{0, 1, 2, 3, 4\}$
   (b) Tabulate the set of ordered pairs.
   (c) Show the mappings by means of two horizontal number lines.

3. Show the mappings determined by $m \rightarrow 2m^2 - 6$ when $M = \{-2, -1, 0, 1, 2, 3, 4\}$.

4. In secret coding, a special mapping is used. Each letter of the alphabet is mapped into a set of numbers or into a set of letters or symbols. For example:

Then the message is sent using numerals instead of letters.

(a) The code given would be too easy to decode. Let us make a more difficult code.



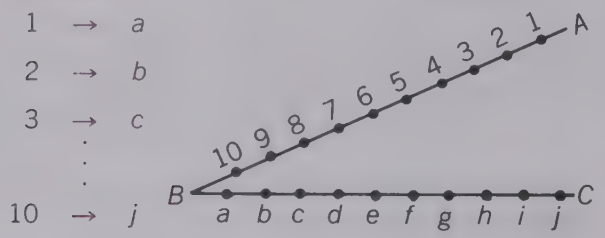
Complete the code then decode this message.
25, 21, 23, 2, 19, 2, 16, 10, 20, 20, 15, 22, 25, 15, 11, 10, 13, 10, 24, 15, 21, 13, 10

(b) Another mapping:

(i) Cross out any letter as it repeats.
(ii) Map the letters of the alphabet on the remaining letters. There should be one letter for each letter of the alphabet.
(iii) Decode: ntoutnusstcutfquyektfcuzbvktipbufq.

Ideas come only once you've trained yourself to look for them

5. Draw angle $ABC = 30°$ with each ray 10 centimetres long. Divide each segment into 10 equal parts. Label each division on AB starting at A with 1, 2, 3, ... , 10. Label the divisions on BC starting at the first division beyond B with $a, b, c, \ldots, j$. Join by straight lines the points indicated by the mapping:

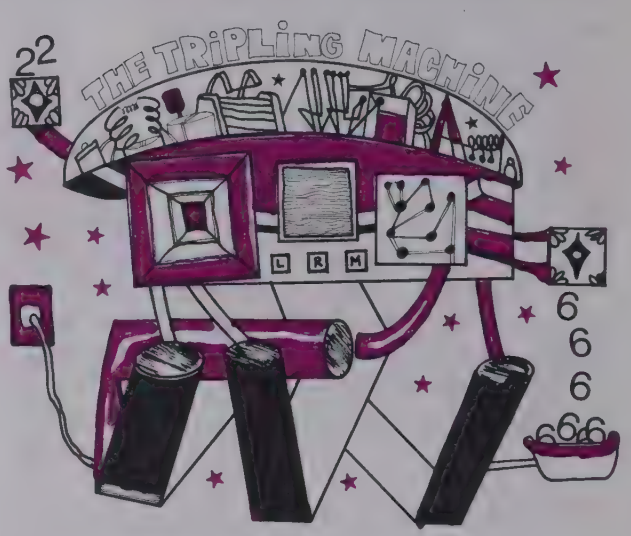


6. Ever wonder why there is a population explosion in Canada? Study the data on this page.
Show this information on one line graph. Notice the postwar *Baby Boom* (World War II).
Why do you think the birth rate was so low in 1931-1940?
Why do you think the birth rate has dropped so rapidly since 1960?
Why do you think the death rate has dropped so rapidly since 1960?

7. Find the range of the mapping
$a \rightarrow -2a - 1$ when $A = \{6, 9, 12, 14, 24\}$.
Draw a mapping diagram.

8. Find the range of the mapping
$m \rightarrow -\frac{1}{3}m - 6$ when $M = \{6, 9, 18, 27, 48, 123\}$.

9. A mapping can be compared to a machine. Some element is put into the machine, the mapping operates on it, and out comes some element. Remember, every element you put in must produce some element and can only produce one element.
Consider the machine to be the *tripling* machine.

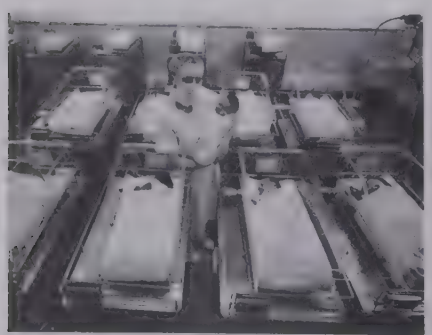


Complete the table of ordered pairs.

input	0	1	2	5	18			
output			6			24	108	52

10. A triangle has a base of 12 inches. Express as a mapping the relation between the altitude of the triangle and its area. What is the area of the triangle if its altitude in inches is
(a) 4 (b) 10 (c) 45 (d) $8\frac{3}{4}$.

11. Express as a mapping the relation between the radius (r) of a circle and the area of the circle. Find the range for each domain.
(a) 7 (b) 14 (c) 30.



Average per thousand		
for Years	*Births*	*Deaths*
1926-30	24.1	11.1
1931-35	21.5	9.8
1936-40	20.5	9.8
1941-45	23.5	9.8
1946-50	27.4	9.3
1951-55	28.0	8.5
1956-60	27.6	8.0
1961-65	25.3	7.7

3·4 Graphs of Relations

Another easy method of showing relations is through the use of graphs.

Example 1

Graph the following relation.

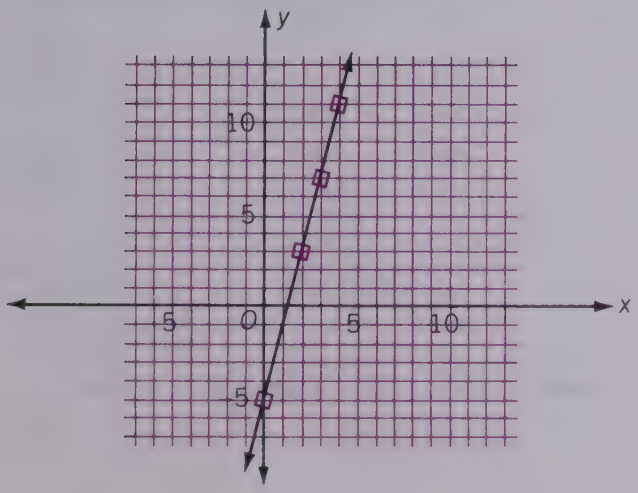
$$U = R \times R, R = \{(x, y) \mid y = 4x - 5\}$$

Solution

The domain and range is the set of real numbers. Since, in the real number line each point on the line can be associated with a number, we cannot tabulate in table form all the ordered pairs for this relation. Therefore we will choose a few ordered pairs, plot these points and then connect the points by lines to show the relation is continuous. The table of ordered pairs of the solution set is

x	0	2	3	4
y	−5	3	7	11

Next we plot the points on the grid of the real number plane. Then we connect the points by a line—in this case a straight line. To show the graph is infinite, place arrows on both ends of the line.



$\{(x, y) \mid y = 4x - 5\}, U = R \times R$

You will recall from the example that each point in a Cartesian plane can be associated with an ordered pair of real numbers. The x-coordinate is called the *ABSCISSA*. The y-coordinate is called the *ORDINATE*. It is by convention we write the x-value first and the y-value second in the ordered pair.

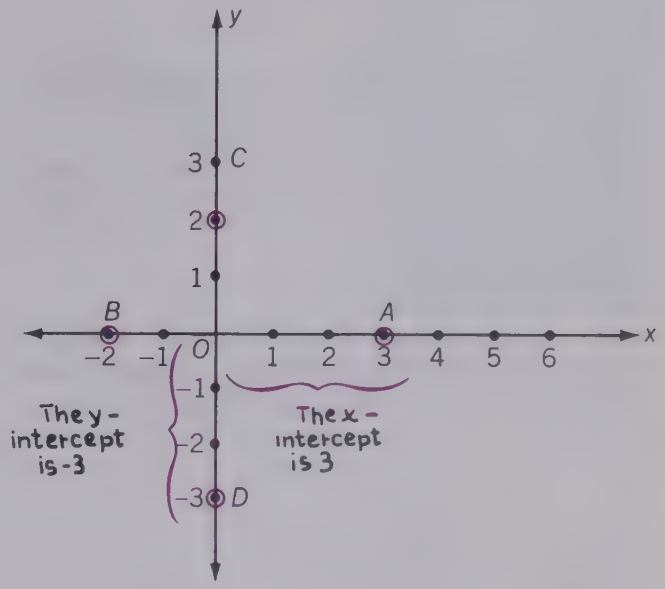


Figure 3-1

In Figure 3-1 the directed distance from the origin 0 to the point *A on the x-axis* is called the *x-intercept*. In this case the x-intercept of point A is 3. The x-intercept of point B is −2. Similarly, the directed distance from the origin 0 to point *C on the y-axis* is the *y-intercept*. The y-intercept of point C is 2 and of the point D is −3.

It is convenient to stipulate from which sets we will select replacements for the variables.

We can write

$$U = R \times R \quad \{(x, y) \in U \mid y = 3x\}$$

or $$\{(x, y) \mid y = 3x, x, y \in R\}$$

The colored portions indicate the replacement set is the reals.

Not all relations are represented by straight line graphs. Consider the following.

Example 2

The area of a square (a) varies with the square of the sides (s). Make a graph of the following relation for the area of a square and the measure of a side of the square.

$$U = R \times R \quad \{(s, a) \mid a = s^2\}$$

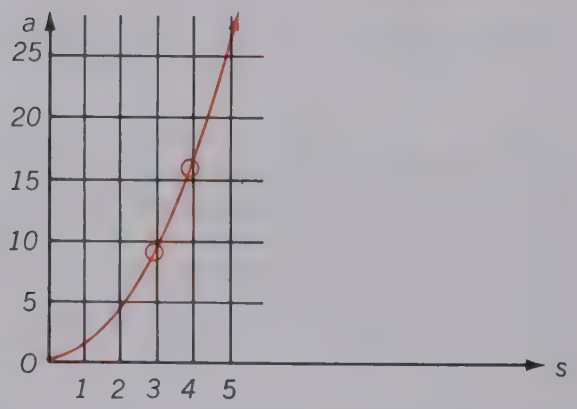
Solution

Since a square cannot have a side of negative length the universe is the set of nonnegative reals.

The table of ordered pairs of the solution set is

s	0	$\frac{1}{2}$	1	2	3	4	5
a	0	$\frac{1}{4}$	1	4	9	16	25

Plot the points. Connect the points with a smooth curved line. Since the size of the square can increase indefinitely we place an arrow on the upper end of the graph.



$$\{(s, a) \mid a = s^2\},\ U = \overline{R^-} \times \overline{R^-}$$

Example 3

In science you learn that as the pressure (p) on a certain mass of gas increases, the volume (v) decreases. This statement can be simplified to this form for a mathematical representation:

$$p = \frac{k}{v} \text{ and if } k = 1,\ p = \frac{1}{v}$$

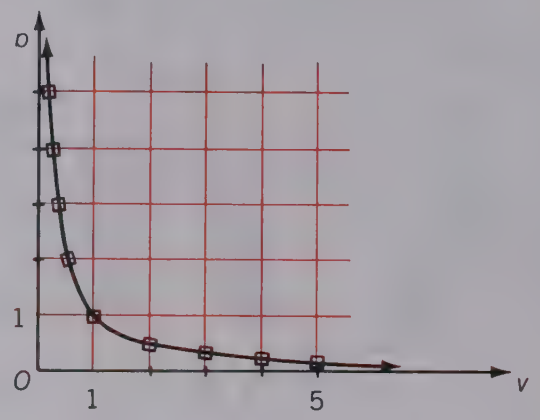
Since pressure and volume are positive quantities we will have $U = R^+ \times R^+$. Graph the relation.

Solution

The table of ordered pairs of the solution set is

v	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4	5
p	4	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

Plot the points and join them with a smooth curved line.



$$\{(v, p) \mid p = \frac{1}{v}\}$$

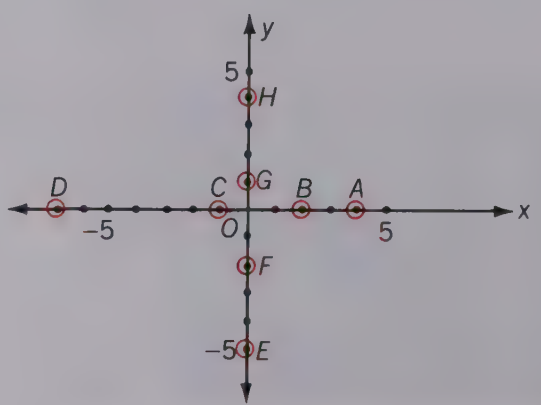
EXERCISE 3·4

1. Name the x-intercept of each of the following points:

   $A \quad B \quad C \quad D.$

   Name the y-intercept of each:

   $E \quad F \quad G \quad H.$



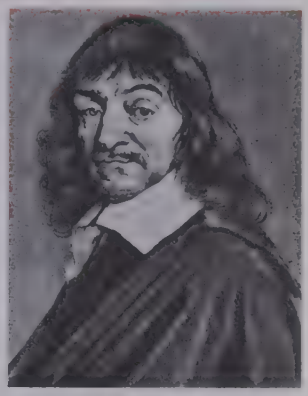
2.

x	0	5	6	8	9	8	6	5	4	−4	−3	−4	−5	−5	−1	0
y	5	5	3	3	2	1	1	0	1	1	0	1	1	3	3	5

Pictures can be sent in code by means of coordinates. Plot the above points, then join the consecutive points by means of a smooth curve. What is this a picture of?

Can you make up a *code* to exchange with your friends?

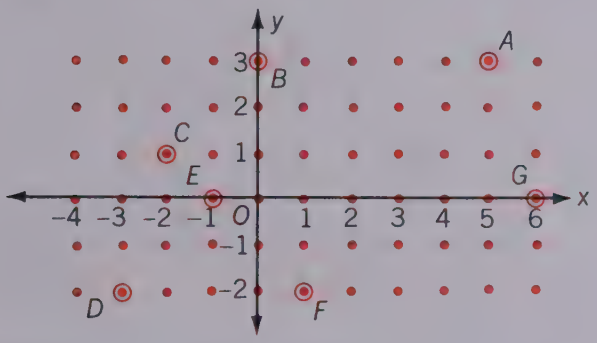
3. René Descartes did considerable work on coordinate planes. Make a report on Descartes in this respect. From where does the word *Cartesian* come?



René Descartes (1596-1650) What did Descartes mean by "Cogito ergo sum"?

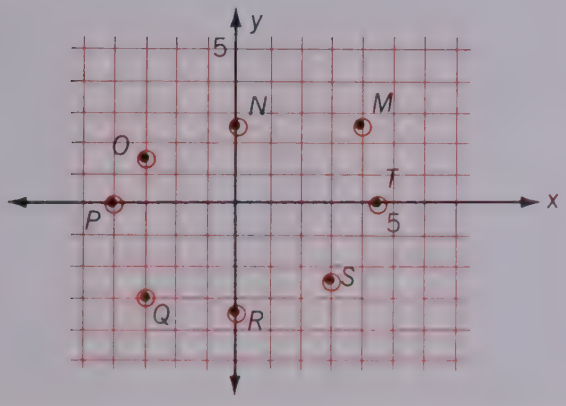
4. Write the ordered pair that corresponds to each point.

(a) *A* (b) *B* (c) *C* (d) *D*
(e) *E* (f) *F* (g) *G*



5. Write the ordered pair that corresponds to each point.

(a) *M* (b) *N* (c) *O* (d) *P*
(e) *Q* (f) *R* (g) *S* (h) *T*



6. Construct a table of ordered pairs of the solution set and plot the graph for each relation.

(a) $U = R \times R, \{(x, y) \mid y = x, x < 6\}$
(b) $U = R \times R, \{(x, y) \mid y = 2x\}$
(c) $U = R \times R, \{(x, y) \mid y = 3x - 4\}$
(d) $U = R \times R, \{(x, y) \mid y = x^2 + 2\}$
(e) $U = R \times R, \{(x, y) \mid y = -x^2 + 6\}$
(f) $U = R \times R, \{(x, y) \mid y = \frac{4}{x}, x \neq 0\}$

7. What do we mean when we say "Plot the graph of the relation"?

Desirable Weights

Weight in Pounds According to Frame (In Indoor Clothing)

	HEIGHT (with shoes on) 1-inch heels Feet	Inches	SMALL FRAME	MEDIUM FRAME	LARGE FRAME
Men of Ages 25 and Over	5	2	112–120	118–129	126–141
	5	3	115–123	121–133	129–144
	5	4	118–126	124–136	132–148
	5	5	121–129	127–139	135–152
	5	6	124–133	130–143	138–156
	5	7	128–137	134–147	142–161
	5	8	132–141	138–152	147–166
	5	9	136–145	142–156	151–170
	5	10	140–150	146–160	155–174
	5	11	144–154	150–165	159–179
	6	0	148–158	154–170	164–184
	6	1	152–162	158–175	168–189
	6	2	156–167	162–180	173–194
	6	3	160–171	167–185	178–199
	6	4	164–175	172–190	182–204

For each size of frame, plot a graph of height and average weight. Do you get a smooth curve?

Projects

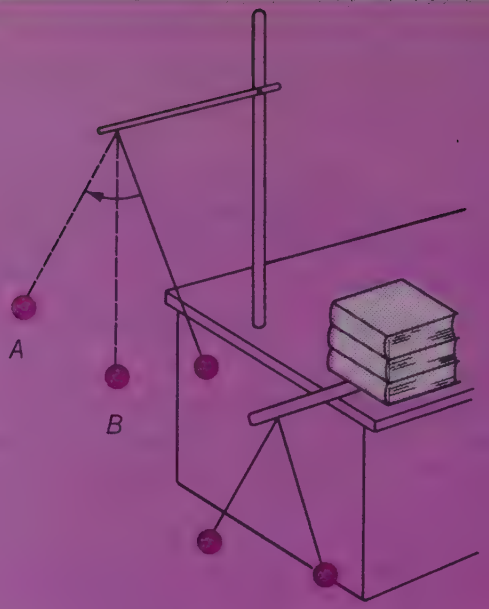
Project 1

Purpose

To determine the relation between the length of a pendulum and the time of one oscillation.

Method

Obtain a stopwatch (from the Science or Physical Education Department), a length of string of 50 inches, a weight of about one-half pound that can be attached to the string, and graph paper. Arrange the apparatus as shown so that the pendulum is free to swing when the length of the pendulum is 48 inches. (If a ruler is used to secure the pendulum, 36 inches should be the length of the string.) Allow the weight to move through an arc of 30° to 40°.



Use the stopwatch to time 20 oscillations. An oscillation is from *A* to *B* and back to *A*.
Repeat for pendulum lengths of 42, 36, 30, 24, 18, 12 inches.

(a) length in inches	48	42	36	30	24	18	12
(b) time for 20 oscillations (seconds)							
(c) time for 1 oscillation (seconds)							

Record in the table above. Plot the information of (a) and (c) on a Cartesian plane. Join the points with a smooth curve. Are there any points that do not seem to fit in the smooth curve? If so, how do you explain the points that do not seem to fit?
What would you suggest in regard to these points?

Project 2

Purpose

To show by graphs the relation between the amount of water and the height of the water in different shaped containers.

Method

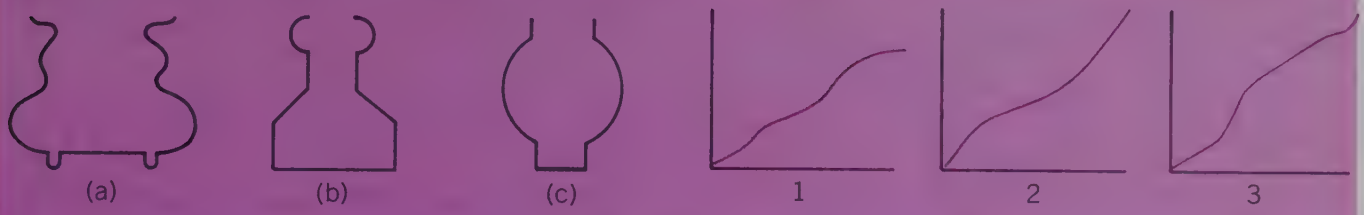
Provide each group of students with the following: graduated cylinder (in millilitres), ruler, water, containers (clear) of odd shapes of 1 pint to 1 quart capacity. (One container per group is sufficient as the containers can be rotated among the groups.)

1. Pour 10 cubic centimetres (cc) of water into your container.
2. Measure the height in millimetres of the water. Record.
3. Pour another 10 cc of water into the container.
4. Measure the height of the water. Record.
5. Continue to pour in 10 cc units of water and to measure the height after each addition until the container is full.
   Record the data.

V – Volume of water	*10*	*20*	*30*	*40*
h – height of water				

6. Draw a graph with the volume of water on the *x*-axis and the height on the *y*-axis. Joint the points on the graph with a smooth curved line.

7. Repeat with several different shaped containers.
8. Following are three graphs and the containers used. Match the graph with the proper container.



Project 3

Purpose

To show the relation of the distance a ball rolls and 1. the steepness of the incline up which it rolls; 2. the steepness of the incline down which it rolls.

Method

Obtain a small steel ball (or large marble) of diameter about $\frac{3}{4}$ inch. Set up the incline plane apparatus as in the diagram. Heavy cardboard works well for the planes. Use masking tape to secure the planes to the table top. Books are suitable to hold up the incline planes though if you have laboratory facilities available, adjustable stands are better. Initial height h of incline B should be a little more than that of incline A. Suggested dimensions are shown.

1. Place the ball on the top of the incline A. Allow the ball to roll down (don't push) and to roll up incline B. Mark the highest point it reached. Record in a table the distance s and the height in centimetres.

h	8	10	12					
s								

Increase the height h of incline B by 2 centimetres and repeat. Continue to increase h by 2 centimetre units six times. Record the data.

Plot the information on a graph using s as the horizontal axis and h as the vertical axis. Draw a smooth curve through the points.

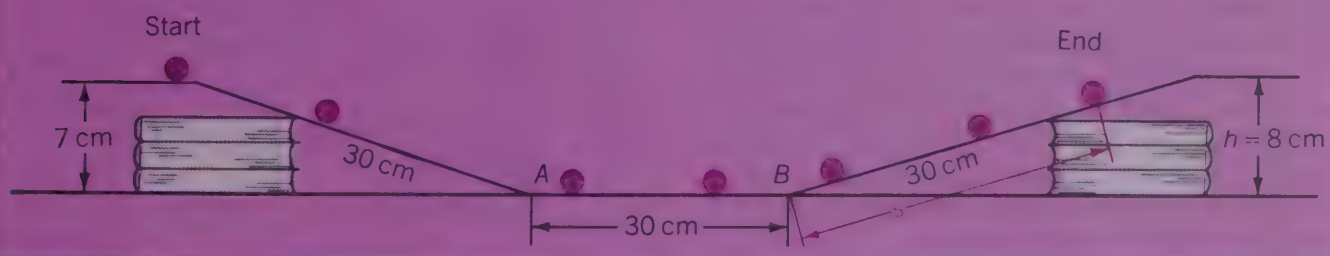
2. Repeat the general procedure in (1) but this time fix the height of incline B at about 12 cm. Start with incline A at a height (h) of 2 cm and increase it each time by 2 cm. Record the distance (s) the ball rolls up incline B.

h	2	4	6						
s									

Plot the graph of the relation in the same manner as in (1).

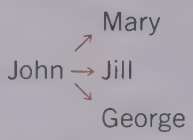
Questions

Does the distance s the ball goes up incline B depend on whether or not the ball goes straight up or on a slant? Do you know for certain or are you guessing? Perhaps you can devise a project to determine the answer.

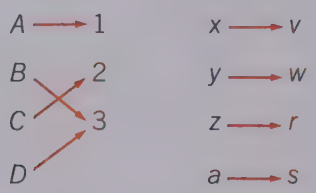


3·5 Functions

Earlier we spoke of the relation *is a brother of*:



In this relation the element in the domain is mapped into more than one element in the range. Hence, we said this is not a mapping. We also showed these two relations:



In these two cases each element of the domain is mapped into exactly one element of the range. These, we said, are mappings. We have another name for those relations in which each member of the domain is mapped into one and only one element in the range. The name is *function*.

> A *function* is a relation in which each element of the domain is mapped into one and only one element of the range.

Let us look at two relations defined by the tabulation of the set of ordered pairs to decide whether or not they define functions.

$A = \{(1,2), (2,3), (3,4), (4,5), (5,6)\}$

$B = \{(2,1), (2,4), (3,5), (3,6), (4,7), (4,8)\}$

Each first member of the ordered pairs in set A is paired with one and only one second member. Since each element of the domain is mapped into exactly one element of the range this relation is a function. In the relation determined by set B, the numeral 2 is paired with 1 and also with 4. Since there is at least one instance in which the first element of the ordered pair is paired with more than one element of the range, this relation is not a function.

Example 1

In the following relations, which are functions and which are not?

$D = \{(x,y) | y = 3x^2 + 2,\ x \in R\}$

$E = \{(x,y) | x = y^2 + 2,\ x \in R\}$

Solution

In the first relation we may tabulate the ordered pairs in set builder notation or we may use the table form:

x	2	1	0	−1	−2
y	14				

When x is replaced by 2, can y have any value other than 14? No.

How many values has y when x is replaced by 1? by 0? by −1? by −2? The answer is *one*.

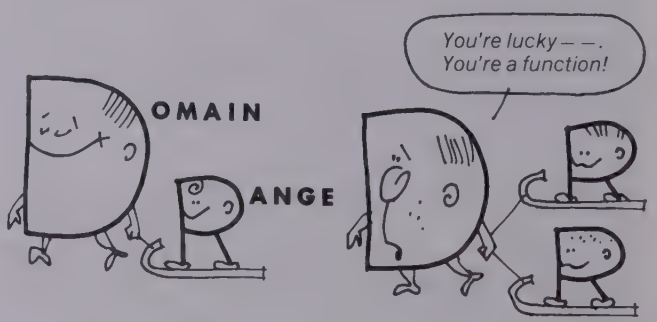
x	2	1	0	−1	−2
y	14	5	2	5	14

Therefore it appears that this relation is a function.

The relation E can be approached in the same manner as we did relation D. When x is replaced by 6, what are the replacements for y? The replacements for y are −2 and +2.

x	6	6
y	−2	+2

Already we have found one element of the domain that is mapped into more than one element in the range. This one instance is sufficient for us to be able to say the relation is *not* a function.



If we graph the relations or if the relations are in graphical form, there is an easy method to decide if a relation is a function or not. Let us graph relations D and E from Example 1.

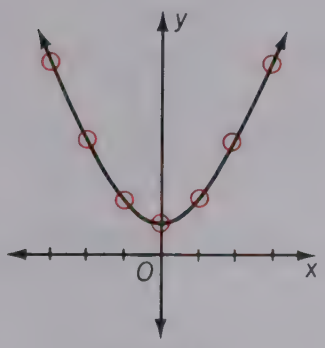


Figure 3-2

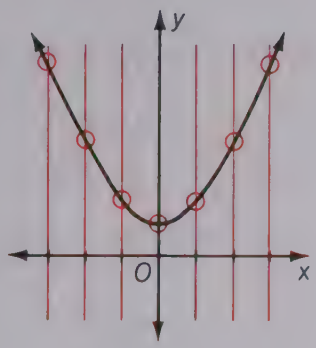


Figure 3-3

$D = \{(x,y) | y = 3x^2 + 2,\ x \in R\}$

x	−3	−2	−1	0	1	2	3
y	11	6	3	2	3	6	11

Now what is it we need to know in order to determine whether or not these are functions? We must know whether or not for some value of x there is more than one value of y. If we sketch in a series of lines parallel to the y-axis as we did on the graph of relation D in Figure 3-3 we are able to see that no vertical line intersects the graph in more than one point. We are not able to envision a vertical line ever meeting the graph at more than one point. Hence, in the solution set for this relation there is at most one value of y for any given value of x. Therefore, the relation is a function.

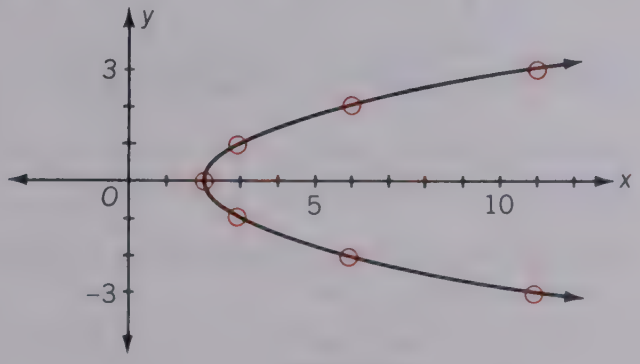


Figure 3-4

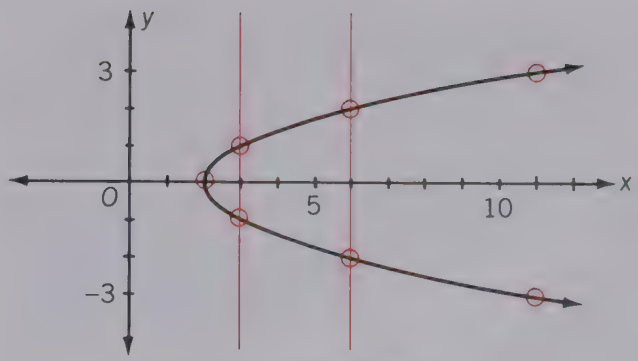


Figure 3-5

$E = \{(x,y) | x = y^2 + 2,\ x \in R\}$

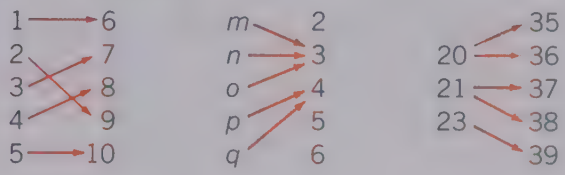
x	11	6	3	2	3	6	11
y	−3	−2	−1	0	1	2	3

Figure 3-4 is the graph of relation E. Sketch in some lines parallel to the y-axis as shown in Figure 3-5. Is there a vertical line which intersects the graph more than once? As an example, what are the y values when $x = 6$? The y values are 2 and −2. The line $x = 6$ intersects the graph twice. There are other vertical lines which intersect the graph twice. Therefore, this relation is not a function.

The *vertical line test* will help you determine whether or not a graph of a relation is also a graph of a function.

EXERCISE 3·5

1. Determine which of the following relations are functions. Explain each. Tabulate the ordered pairs in each function.



2. Let $R = \{(2,4), (4,6), (12,14), (18,16)\}$.
   (a) The domain of R is _____ .
   (b) The range of R is _____ .
   (c) Each element of the domain is paired with _____ element(s) of the range.
   (d) Is R a function?

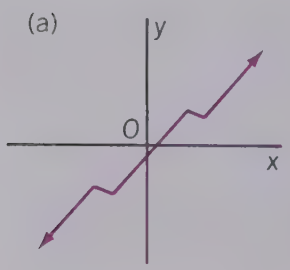
3. Let $M = \{(100,1), (95,2), (90,3), (85,4), (75,1), (70,2)\}$.
   (a) The domain of M is _____ .
   (b) The range of M is _____ .
   (c) Each element of the domain is paired with, at most, _____ element(s) in the range.
   (d) Is M a function?

4. Let $Z = \{(9,3), (16,4), (25,-5), (36,-6), (9,-3), (4,-2), (1,-1)\}$.
   (a) The domain of Z is _____ .
   (b) The range of Z is _____ .
   (c) Relation Z (is, is not) a function because _____ .

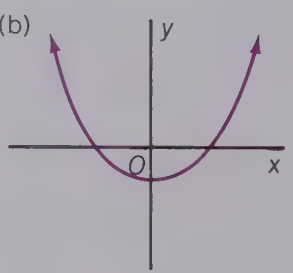
5. Determine which of the following relations are functions. Explain each decision.
   (a) $\{(2,3), (\frac{4}{2},5), (\frac{6}{2},7), (\frac{8}{2},9)\}$
   (b) $\{(4,7), (5,9), (7,11), (\frac{8}{2},\frac{14}{2})\}$
   (c) $\{(12,30), (24,36), (36,42), (\frac{48}{1},42), (\frac{60}{1},36), (72,30)\}$
   (d) $\{(6,4), (8,1), (10,-2), (33\frac{1}{2},\pi), (8,22), (11,17)\}$
   (e) $\{(\pi,2), (\sqrt{2},7), (3\sqrt{6},\pi), (7,\sqrt{2}), (3\sqrt{6},2\pi), (\sqrt{7},2)\}$

6. Use the *vertical line test* to determine which of the following are functions.

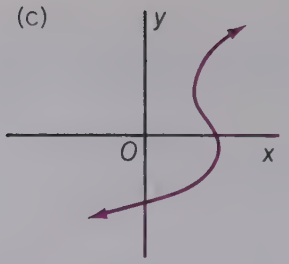
(a)



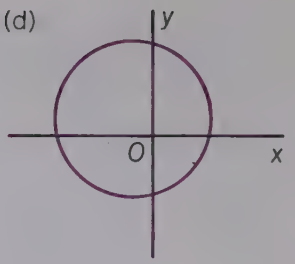
(b)



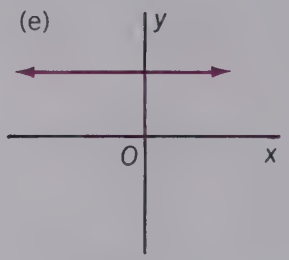
(c)



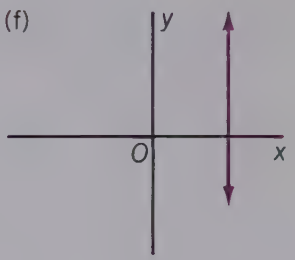
(d)



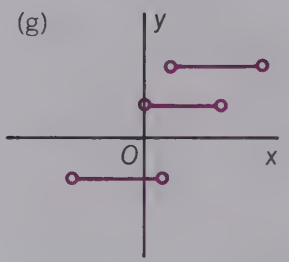
(e)



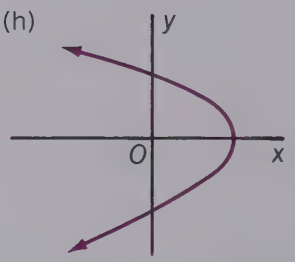
(f)



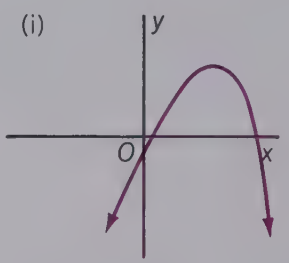
(g)



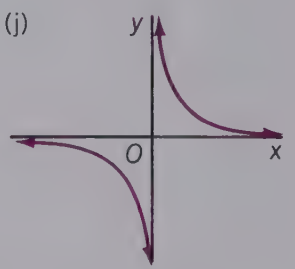
(h)



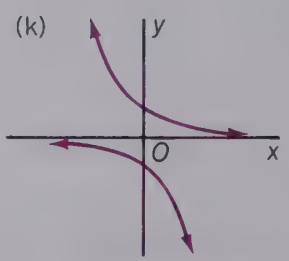
(i)



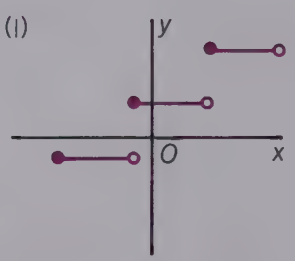
(j)



(k)



(l)



7. Which of the following are functions and which are not? Explain your answer.
   (a) The mark on a mathematics test → the name of student receiving the mark.

(b) The name of a person → his telephone number.

(c) A telephone number → name of person using the telephone.

(d) Measure of the area of a rectangle → rectangle.

(e) Rectangle → the measure of the area of the rectangle.

(f) Birthdate of person → name of student.

(g) Name of student in your class → his birth date.

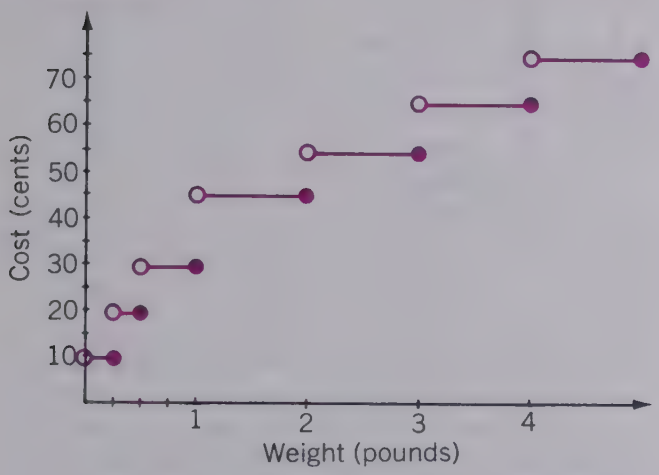
(h) Social Insurance Number → name of employee.

(i) Name of employee → Social Insurance number.

(j) Measure of the area of a square → square.

(k) Title of this text → author.

8. The cost of sending parcels by mail is shown in the graph.



(a) What is the postage for a 2-ounce parcel? a $\frac{1}{2}$-pound parcel? a $1\frac{1}{2}$-pound parcel? a 4-pound parcel?

(b) What do you think is the meaning of the open and shaded dots at the end of each horizontal segment?

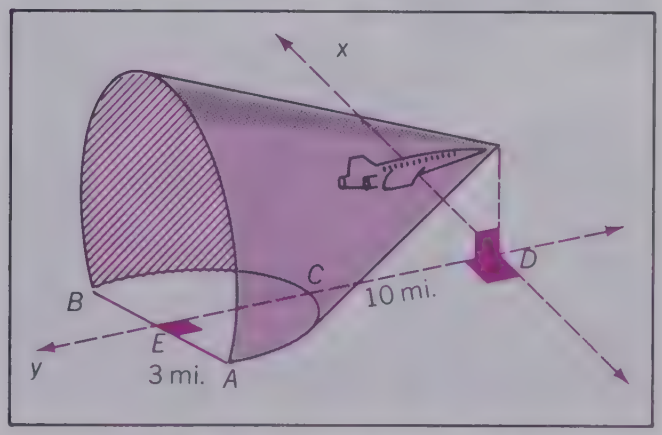
(c) Is the relation of "*weight* → *cost*" a function? Why?

9. A sonic boom is experienced when an airplane travels faster than sound (supersonic) The wave

front, set up by the airplane, is in the shape of a cone. The pattern of the sonic boom on the ground is that of a parabola. If the parabola can be represented by

$$y = 2x^2 + 10$$

where both y and x are in miles and the 10 represents $\overline{CD}$, the distance the plane has past the point where the sonic boom strikes the ground directly under the flight path.



How far is the plane past the point E when the sonic boom hits point A which is 3 miles from point E on the flight path?

3·6 Shorthand

We often put in words that which can be expressed by mathematical symbols. For instance we might say "Add 5 to each of the given numbers." We can write this as "$x + 5$" where it is understood that x will be replaced by the given numbers, one at a time. We can write this as a mapping

$$x \to x + 5.$$

Example 1

Write each of these functions in the form

$$x \to \blacksquare.$$

(a) Divide by 3.
(b) Multiply by 2 and add 1.

Solution

(a) $x \to \frac{x}{3}$

(b) $x \to 2x + 1$

Relations are often easier to see when illustrated on a graph. This student in a technical school studies the relation of certain substances and wavelengths of their spectra.

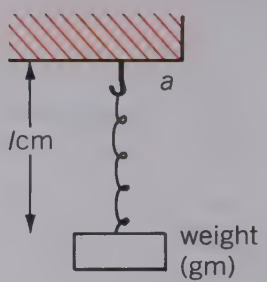
EXERCISE 3·6

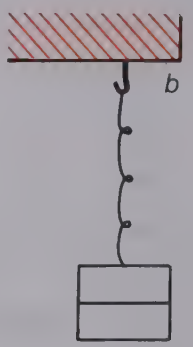
1. Write each of the following relations in the form

$$x \to \blacksquare.$$

(a) Multiply by 3 .
(b) Subtract 3.
(c) Multiply by 5 and add 13.
(d) Add 13 and multiply by 5.
(e) Take the principle square root.
(f) Square and add π.
(g) Square, add 5 and multiply by 3.

2. Obtain a spring (or several elastic bands), fix it at one end and hang different weights at the other end. Record the length of the spring each time a different weight is fixed to the lower end. Record the information in a table and plot it on a graph. Draw a curve through the points on the graph. Is this a linear function?





3. A sonic boom is a shock wave accompanied by a sudden increase in air pressure – followed by an equal decrease from the normal. This increase may be from 0.5 to 2 pounds per square inch. If the normal air pressure is 14.3 pounds per square inch, give the maximum and minimum air pressure accompanying the boom if
(a) the increase is a minimum
(b) the increase is a maximum.

Can you represent this as a relation?

What is the pressure on a living room window during a sonic boom caused by the increased air pressure if the increase is a
(a) minimum? (b) maximum?

Do you think either of these would break windows?

3·7 Graphing Functions

This section consists of a number of exercises followed by questions. In some cases the questions are designed to help you do the exercises. In other cases the questions develop new ideas. Copy and complete each incomplete statement.

I. Set up a table of values, and then draw a graph of the relation defined by each of the questions.

(a) $2x + y = 3$ (b) $3x + 6 = 2y$

(c) $4y - 3 = x$ (d) $y = 0x + 4$

(i) What do the four graphs have in common? (What do the graphs look like?)

(ii) A straight line graph is called a *linear* graph. Since each equation yielded a linear graph, we call each of these equations a _____ equation.

(iii) What is the exponent of the x in each equation?

(iv) What is the exponent of the y in each equation?

(v) Equations with two variables each with the exponent '1' are represented by _____ line graphs? These equations are _____ degree equations in two variables.

(vi) Which of the relations are functions?

II. On a Cartesian plane, locate and mark the two points (2, 4) and (8, 6).

(a) Draw a straight line containing the two points.

(b) How many distinct lines can you draw through the two points?

(c) How many points are required to determine a straight line?

(d) If you are asked to draw the graph of a relation defined by a first degree equation in two variables, you know it will be a _____ line graph. How many points must you know in order to draw this line?

III. Given the relation

$$\{(x, y) \mid 2x + 3y = 6\}$$

answer these questions then draw the graph of the relation.

(a) What kind of a graph will this be?

(b) How many ordered pairs in the solution set *must* you know in order to draw the graph?

(c) Complete as much of this table as necessary to enable you to draw the graph of this relation.

x							
y							

(d) Draw the graph of the relation.

(e) Is the relation a function?

(f) Many scholars maintain a student should always use three points to draw the graph of a linear function. Can you think of reasons the third point should be used? Do you think you should use three points?

IV. The above information may be summarized in the following statement. Copy and complete the statement by placing the correct word or words in the blanks.

> An equation involving two variables each with the exponent 1, i.e., $4x + 2y = 5$, is called a _____ degree equation or a _____ equation. The graph of the relation defined by such an equation is called a _____ graph or a _____ line graph. To draw a graph of the relation requires at least _____ ordered pairs that satisfy the relation and which identify points on the Cartesian plane since only _____ points are required to determine a straight line. A _____ point is often used as a _____ to make certain no _____ have been made.

V. The following steps will illustrate another method of graphing linear functions. Can you identify what is different about this method?

$$\{(x, y) \mid 2x + 3y = 6\}$$

(a) If $y = 0$ what value of x satisfies the function? The point with the coordinates (?, 0) is in the graph.

(b) Locate this point on the Cartesian plane. Label this point A.

(c) What is the x-intercept of point A?

(d) If $x = 0$ what value of y satisfies the function? The point with the coordinates (0, ?) is in the graph.

(e) Locate this point on the Cartesian plane in (b). Label this point B.

(f) What is the y-intercept of point B?

(g) Do we have enough information to draw the graph now? Why?

(h) Complete this table from the above information. Draw the graph.

x		0
y	0	

(i) What suggestion would you make now to ensure that the graph is correct? Follow your suggestion.

(j) In this method we used the x-intercept and the y-intercept to locate the two points which determine the linear graph.
What name would you give this method?

(k) Copy and complete this statement:

Let us summarize. To graph a linear function such as

$$\{(x, y) \mid 3y + x = 9\}$$

note the following:

1. The x-intercept determines the point whose ordinate is ____. The x-intercept is _____. The ordered pair for this point is _____.

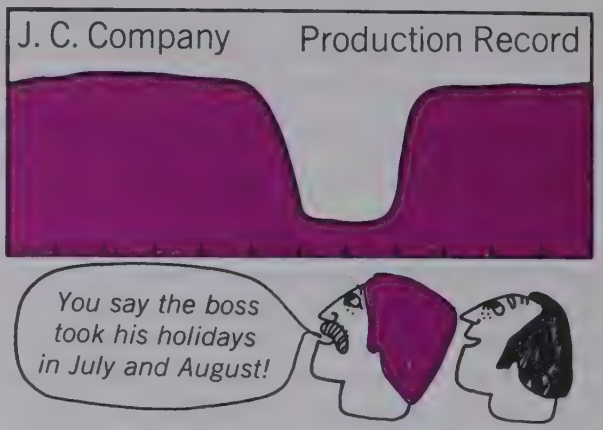
2. The y-intercept determines the point whose abscissa is ____. The y-intercept is _____. This second point has the coordinates (?,?)

3. Locate these two points on a Cartesian plane. Draw the line determined by the two points.

4. It is wise to arbitrarily select a third point as a_____to guard against____.

How does this method differ from the previous method?

Since graphs tell a story simply, they are used in cartoons.



EXERCISE 3·7

1. Use the "two-point" method to graph these functions. Complete the table in each case. Use the third point as a *check*.

(a) $h = \{(x, y) \mid 3x - 2y = 12\}$

x	0	4	2
y			

(b) $i = \{(x, y) \mid y = -5x + 4\}$

x	1	3	5
y			

(c) $k = \{(x, y) \mid x = 3 - 2y\}$

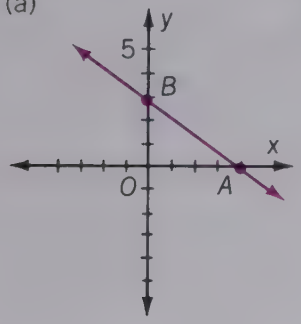
x	−3	7	11
y			

(d) $l = \{(x, f(x)) \mid f(x) = \frac{2x + 4}{3}\}$

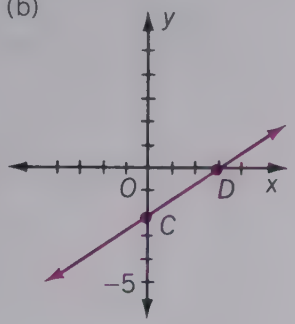
x	−2	1	4
$f(x)$			

2. For each of the following graphs give, where possible, the
(i) x-intercept (ii) y-intercept.

(a)



(b)



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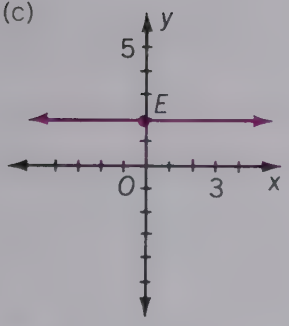
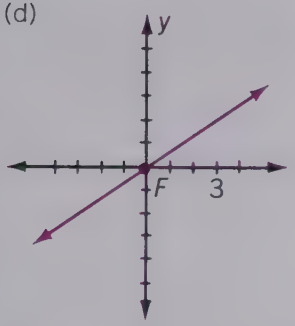
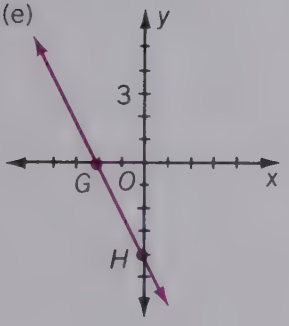


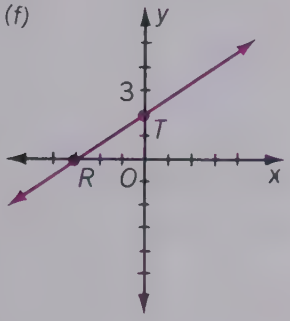
(d)



(e)



(f)



3. Refer to the graphs in Exercise 2. Give the coordinates of the following points.

A B C D E F G H

4. Draw the graphs of the following functions by the "x and y-intercept method". (See V above)

(a) $\{(x, y) \mid y = -2x + 8\}$

(b) $\{(x, y) \mid y - 3x = 6\}$

(c) $\{(x, y) \mid 4x - 3y = 12\}$

(d) $\{(x, y) \mid 5y - 6x = 30\}$

5. Draw the graphs of the following functions.

(a) $\{(x, y) \mid y = 3x\}$

(b) $\{(x, y) \mid y = 4\}$

(c) $\{(x, y) \mid 8x - 7y = -56\}$

Do you seem to be going nowhere fast? Do you have your objectives clearly defined? Where do you really want to go?

3·8 Equivalent Functions

Let us graph these two functions:

$a = \{(x, y) \mid 2x + 3y = 6\}$

$b = \{(x, y) \mid y = -\frac{2}{3}x + 2\}$

Using the x and y-intercept method for a we have

x	3	0
y	0	2

The graph is shown in Figure 3-6.

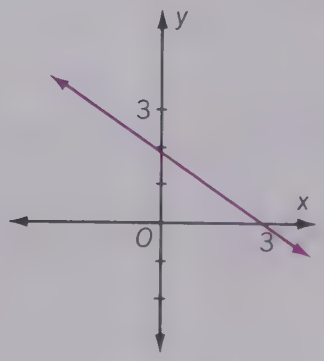


Figure 3-6

The table for b is

x	3	0
y	0	2

The graph is shown in Figure 3-7.

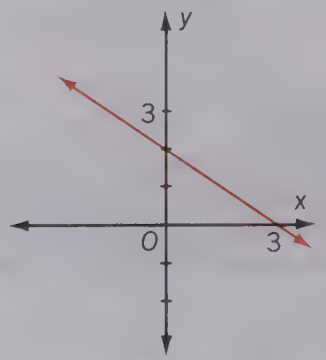


Figure 3-7

It appears the graph of a is the same as the graph of b. If this is so, then the solution set of function a is identical to the solution set of function b. It would seem the equations must be equivalent. Can we prove $2x + 3y = 6$ is equivalent to $y = -\frac{2}{3}x + 2$?

	Authority
$2x + 3y = 6$	
$2x - 2x + 3y = 6 - 2x$	_____
$0 + 3y = 6 - 2x$	_____
$3y = 6 - 2x$	_____
$\frac{1}{3} \times 3y = \frac{1}{3}(6 - 2x)$	_____
$y = 2 - \frac{2}{3}x$	_____
$y = -\frac{2}{3}x + 2$	_____

We have just shown that $2x + 3y = 6$ is equivalent to $y = -\frac{2}{3}x + 2$. We say $y = -\frac{2}{3}x + 2$ is the y-form of the linear function $2x + 3y = 6$. If, for generality, we represent the constant $-\frac{2}{3}$ by m and the constant 2 by b the y-form is

$$y = mx + b,\ m \neq 0$$

EXERCISE 3·8

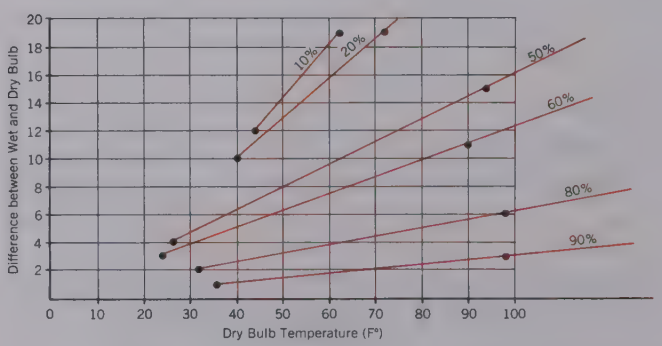
1. Answer the following questions. Support your answers.
   (a) Is $2x + 3y = 6$ equivalent to $2x = 6 - 3y$?
   (b) Is $2x + 3y = 6$ equivalent to $3y = 6 - 2x$?
   (c) Is $2x - 3y = 0$ equivalent to $2x = 3y$?
   (d) Is $ax + by = c$ (a, b, c are constants) equivalent to $ax = c - by$? to $by = c - ax$?
   (e) Is $2x = 3y + 8$ equivalent to $x = \frac{2}{3}y - 4$?
   (f) Is $4x = 8 - 12y$ equivalent to $x = 2 - 3y$?

The following are written in the y-form. State the values of 'm' and 'b' in each.

2. $y = 3x - 4$
3. $y = \frac{41}{2}x + \pi$
4. $y = -3x$
5. $y = \sqrt{3}x - \sqrt{3}$
6. $y = -\sqrt[3]{6} - \frac{4}{3}$
7. $y = \sqrt{5}x$

Rewrite the following in the y-form.

8. $y + x = 4$
9. $x - y = 14$
10. $y - 3x = 5$
11. $2x - y = 3$
12. $y - 2x - \frac{5}{2} = 0$
13. $y + \frac{7}{2} + \frac{3}{2}x = 0$
14. $4 - y + x = 3$
15. $3x - 6y = 2$
16. $y + 2x = 7$
17. $2y + 3 - 2x = 6$
18. $5x = 4y + 7$
19. $3y - 4x = 6 - 2y - 3x$



Relative humidity—a relation expressed in terms of the difference between the dry bulb and wet bulb temperatures and the dry bulb temperatures can be shown by straight lines on this graph.

3·9 Graphing by the "Y-intercept, Slope" Method

There are many methods used to draw graphs. We will study one more since it will have a bearing on later work. Follow through the examples carefully answering the questions.

Example 1

Draw the graph of the function

$$\{(x,y)|y = 2x\}.$$

Solution

In this function, the replacement for y is always 2 times the replacement for x. We can call this a *multiply by two* function or just simply a *multiplier function*. To graph this function we start with the point whose coordinates are (0,0). Label this point A. Does the pair (0,0) satisfy the function? When we increase the value of x from 0 to 1, the value of the function increases from 0 to 2.

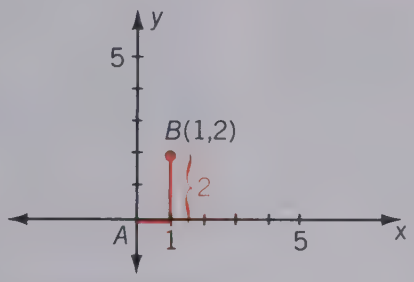


Figure (a)

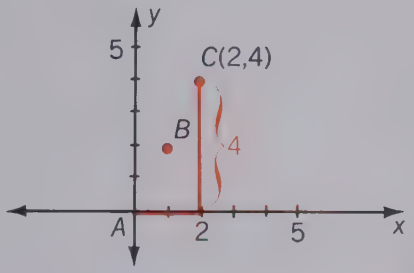


Figure (b)

To graph the second point, place your pencil point on point A, move horizontally 1 unit (to the right) then move up vertically 2 units. This point has the coordinates (1,2). Label this point B. See Figure (a). To graph the third point, place your pencil on point A, move horizontally to the right 2 units then move up 4 units. The coordinates of this point are (2,4). Label this point C. See Figure (b). We now have three points, A, B, and C. Figure (c) is the graph of the function $y = 2x$. Check to see that the y-coordinate is always twice the x-coordinate.

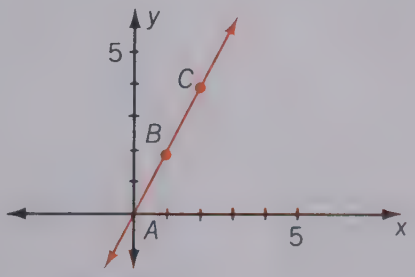
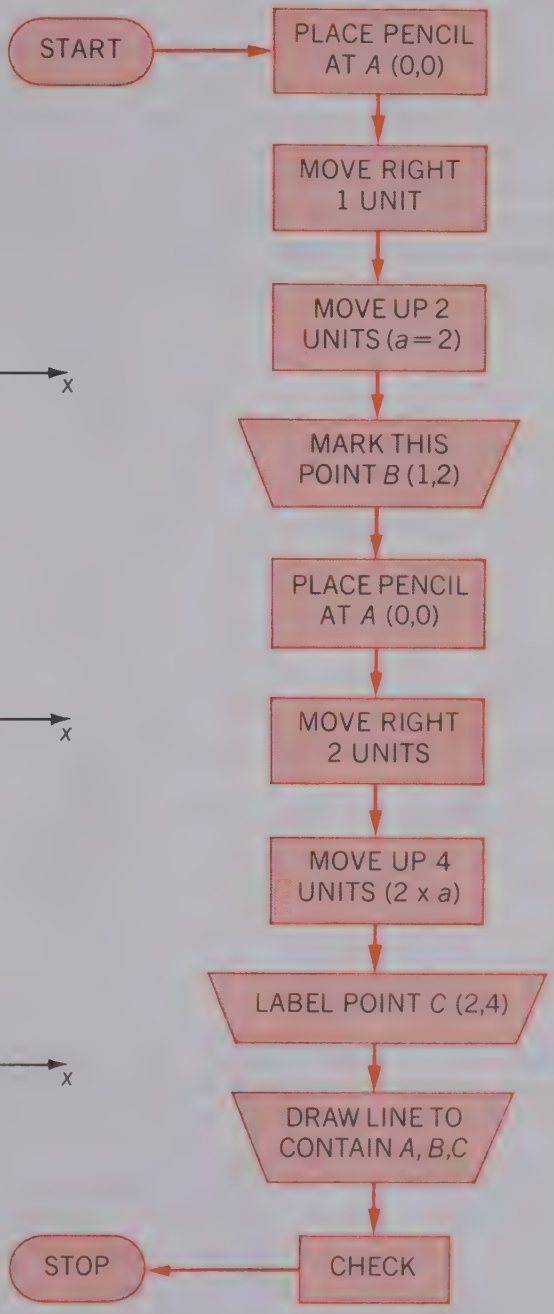


Figure (c)


START
PLACE PENCIL AT A (0,0)
MOVE RIGHT 1 UNIT
MOVE UP 2 UNITS (a = 2)
MARK THIS POINT B (1,2)
PLACE PENCIL AT A (0,0)
MOVE RIGHT 2 UNITS
MOVE UP 4 UNITS (2 x a)
LABEL POINT C (2,4)
DRAW LINE TO CONTAIN A, B, C
CHECK
STOP

Example 2

Draw the graph of the function $\{(x,y)|y = -3x\}$

Solution

On this function the replacement for y is always -3 times the replacement for x. This is a *multiplier function*. To graph this function we start as we did in Example 1, at the point whose coordinates are (0,0). Label this point A. When we *increase* x from 0 to 1, the function *decreases* from 0 to -3; the function decreases by 3 units when x increases by 1 unit. See Figure (a). To graph the second point, place your pencil point on point A, move horizontally to the right 1 unit, then move *down* 3 units. Label this point B.

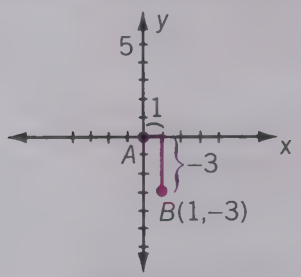


Figure (a)

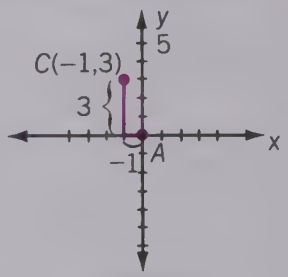


Figure (b)

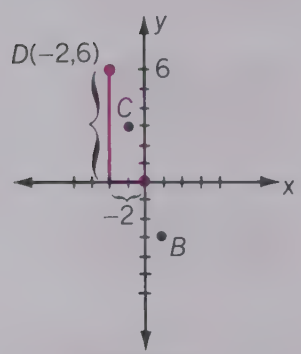


Figure (c)

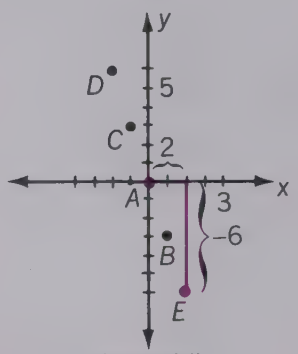


Figure (d)

Point B has the coordinates (1,−3). To graph a third point, place your pencil on point A, move horizontally to the *left* 1 unit then move *up* 3 units. The function *increases* by 3 units when x *decreases* 1 unit. Label this point C. Point C has the coordinates (?,?) Let us graph another point. Place your pencil point on point A, move horizontally to the left 2 units, then move _____6 units. The coordinates of this point are (?,?). Label this point D. Try another point. Move horizontally from point A, 2 units to the right, then move_____ units_____. Label this point E. The coordinates of this point are (?,?). Draw a line to contain points A, B, C, D, E. Figure (e) shows the graph of the function.

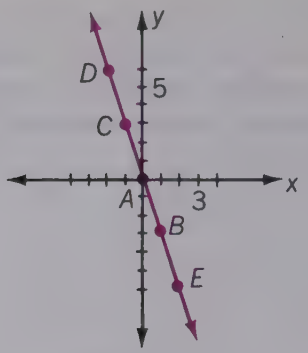


Figure (e)

EXERCISE 3·9

Copy and complete the statements in each question

1. In the function $\{(x,y)|y = 5x\}$, if the x-value increases by 1 unit, then the y-value _____ by __ units; if the x-value increases by 3 units, then the y-value _____ by __ units; if the x-value decreases by 2 units, then the y-value _____ by __ units. When x has a value of 0, the y-value is _____. This is a (multiplier, constant) function.

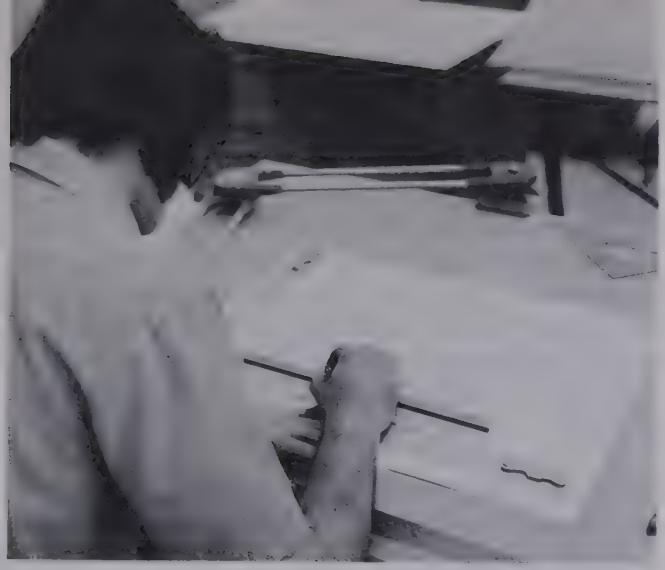
2. In the function $\{(x,y)|y = -3x\}$ when x has a value of 0, the y-value is _____. If the x-value increases by 1 unit, the y-value _____ by __ units; if the x-value increases by 4 units, the y-value _____ by __ units; if the x-value decreases by 3 units, the y-value is _____. This is a (multiplier, constant) function.

3. In the function $\{(x,y)|y = 4\}$ when x has a value of 0, the y-value is _____. If the x-value increases by 2 units the y-value _____; if the x-value decreases by 4 units, the y-value _____. This is a (multiplier, constant) function.

4. Draw a flow chart for Example 2.

5. In the manner described in this section, draw the graphs of the following relations.

   (a) $\{(x,y)|y = 3x\}$ (b) $\{(x,y)|y = -2x\}$
   (c) $\{(x,y)|y = -4x\}$ (d) $\{(x,y)|y = 5x\}$



A table for $y = 3x$ is:

x	0	1	2
y	0	3	6

A table for $y = 2$ is:

x	0	1	2
y	2	2	2

A table for $y = 3x + 2$ is:

x	0	1	2
y	2	5	8

We can obtain this last table by adding the two y values in each case, as shown. The constant function increases the multiplier function by the constant amount 2. Study these figures to see how this is shown by graphs.

3·10 Y-Intercept, Slope Method Continued

Let us graph the constant function

$$\{(x,y)|y = 2\}$$

This may be written

$$\{(x,y)|y = 0x + 2\}$$

The table of values is

x	0	1	2
y	2	2	2

The graph is shown in Figure 3-8.

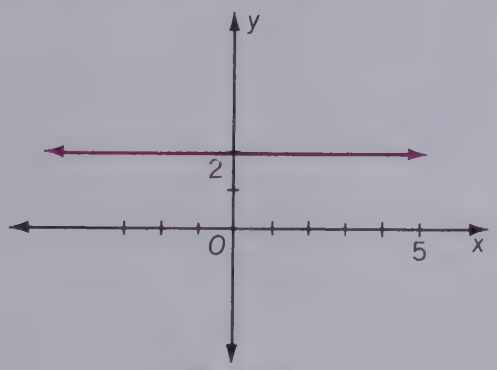


Figure 3-8

Let us now combine two functions: the multiplier function, $y = 3x$, and the constant function, $y = 2$.

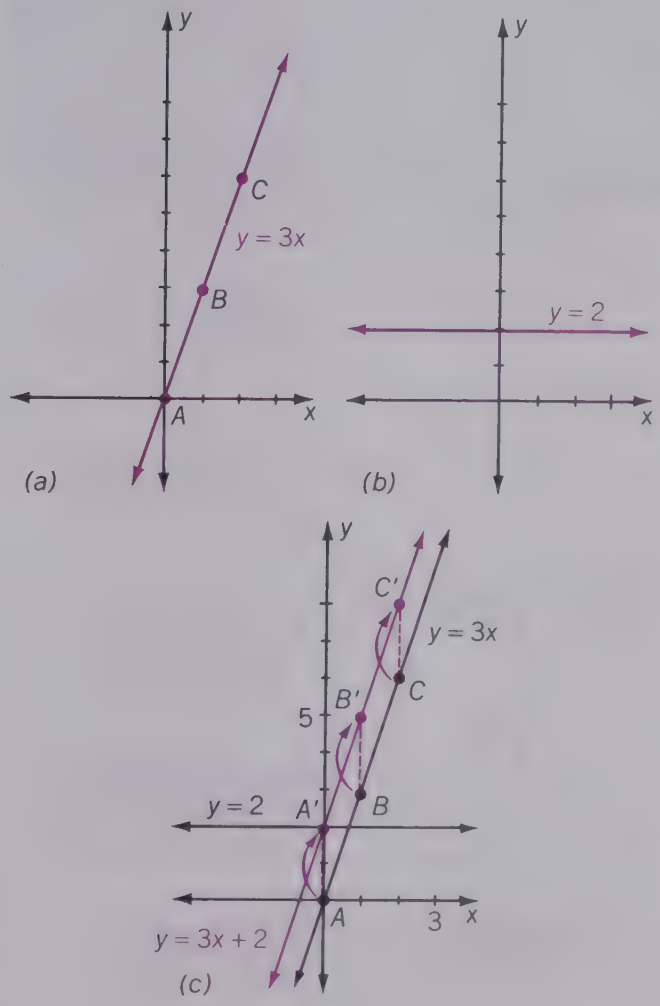


Figure 3-9

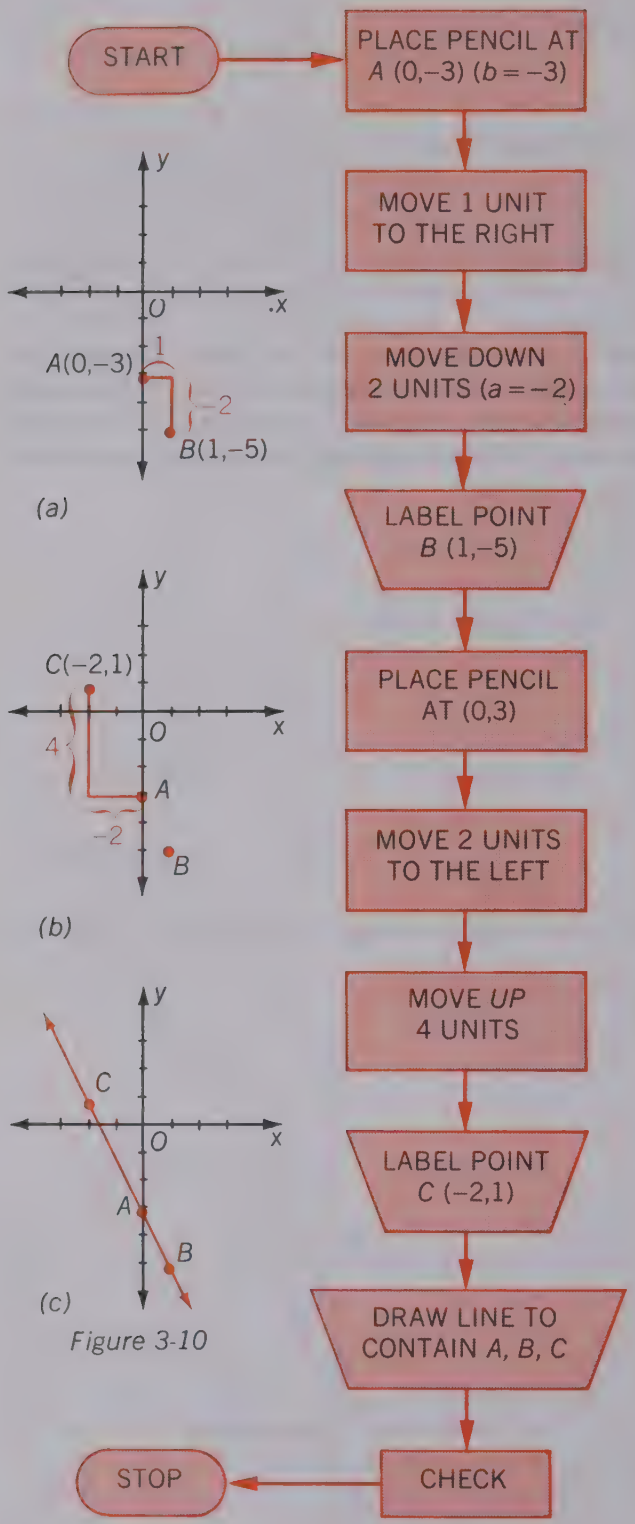


Figure 3-10

The multiplier function has been raised 2 units upward. Point A in Figure 3-9(a) has been raised 2 units to point A' in Figure (c). The y-intercept is 2. Point B has been raised 2 units to B'. Notice in Figure 3-9(c) that we can reach point B' from A' by moving 1 unit to the right and 3 units up. From point A' we can reach point C' by moving to the right 2 units then moving up 6 units. This compares very closely to the procedure used in Example 1 with the only difference being that we start at (0,2) rather than (0,0). Each point on the graph of $y = 3x$ has been raised vertically 2 units.

Let us use the information we have gained to graph the function $\{(x,y)|y = -2x - 3\}$. The starting point is the point with coordinates (0,–3). The y-intercept is –3. Label this point A. Place your pencil on point A, move 1 unit to the right then move down 2 units to graph the point B. (The multiplier is –2). The coordinates of point B are (1,–5). Place your pencil on point A, move 2 units to the left, then move up 4 units to graph point C. Draw a line to contain points B, A, C. Figure 3-10 shows the various moves and the completed graph. This method is known as the "y-intercept, slope" method. We will learn about slope in the next section.

EXERCISE 3·10

1. Copy and complete:

   In the function $\{(x,y)|y = 4x - 5\}$ if the x-value increases by 1 unit, the y-value _____ by _____ units; if the x-value decreases by 2 units, the y-value _____ by _____ units; if the x-value decreases by 4 units, the y-value _____ by _____ units. The y-intercept of the graph of this function is _____. In graphing this function our starting point would have the coordinates (0,?). Label this point M. We would then move our pencil horizontally 2 units to the right and ___ units up. The coordinates of this point (N) are (?,?) A line is then drawn to contain the points M and N. This line is the graph of the function. Draw the graph.

2. In the manner described in this section, draw the graphs of the following relations. Draw a flow chart for (a).
   (a) $\{(x,y)|y = -5x + 2\}$
   (b) $\{(x,y)|y = \frac{1}{2}x + 3\}$
   (c) $\{(x,y)|y = -3x + 5\}$

3. Assume the value of coal produced in Canada yearly to be $73,450,000 and that 30% of this is from strip mining. If the government places a 1.5 percent surcharge on the strip mine production to cover cost of restoration of the land, what sum of money would be available for this purpose?

   If a crew of 25 people and equipment cost $2,250 a day, how many days a year could this crew work on the money for restoration?

Strip mines can be restored to usable grazing land.

This strip mine has not been restored. Discuss its value to a community in comparison to that area pictured in the first column.

3·11 Inequations

Not all mathematical relations are equalities. In everyday discussions we use statements of inequalities:

John has less than $10.00

May is more than 14 years old.

Stella drives less than 30 miles per hour.

Jake has more than five guppies.

In mathematical language we can write these statements:

$$J < 10$$
$$M > 14$$
$$S < 30$$
$$J > 5$$

Let us look at the graph of

$$\{(x, y) \mid y = x + 3, x, y \in R\}$$

in Figure 3-11.

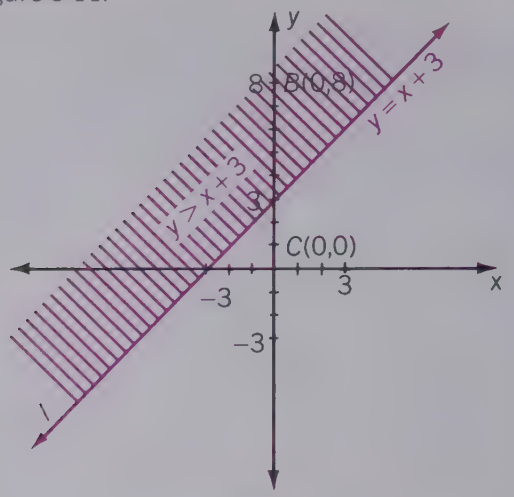


Figure 3-11

For each point on the line l we know the y-value is equal to $x + 3$. But what about the points in the half-plane above the line? Consider the coordinates of point B: $x = 0$, $y = 8$. Is $y = x + 3$ or $y < x + 3$ or $y > x + 3$? By making replacements we have

$$8 \boxed{?} 0 + 3$$

and so we know $y > x + 3$ at point B. Test other points above line l.

Now consider points in the half-plane below line l. Consider point C: $x = 0$, $y = 0$. Is $y = x + 3$ or $y > x + 3$ or $y < x + 3$? Making replacements we have

$$0 \boxed{?} 0 + 3$$

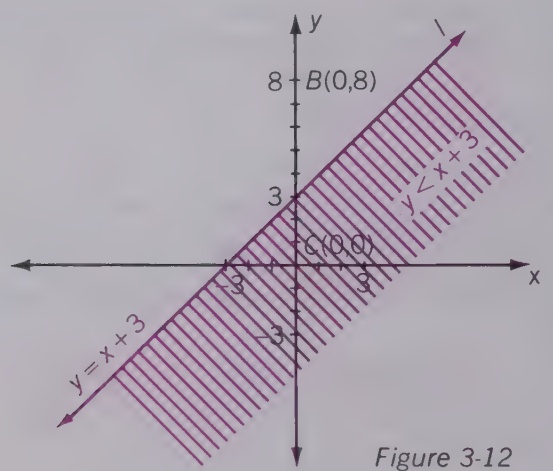


Figure 3-12

Hence at point C, $y < x + 3$. Test other points in the half-plane below line l.

From one example we conclude:

> Points above the graph of the relation $y = ax + b$ satisfy the relation
>
> $$y > ax + b.$$

> Points below the graph of the relation $y = ax + b$ satisfy the relation
>
> $$y < ax + b.$$

EXERCISE 3·11

1. Make the graph of $\{(x, y) \mid y = -2x\}$, $U = R \times R$. Shade in that portion of the plane in which the y-value of any point is less than $-2x$.

2. For each open sentence in part (i) make a table of ordered pairs of the solution set, and draw the graph.

   For each open sentence in part (ii) shade in the appropriate half-plane of the graph drawn for part (i).

   (a) $U = R \times R$
   - (i) $\{(x, y) \mid y = 2x + 1\}$
   - (ii) $\{(x, y) \mid y < 2x + 1\}$

   (b) $U = R \times R$
   - (i) $\{(x, y) \mid y = -x + 2\}$
   - (ii) $\{(x, y) \mid y > -x + 2\}$

   (c) $U = R \times R$
   - (i) $\{(x, y) \mid y = 3x - 2\}$
   - (ii) $\{(x, y) \mid y < 3x - 2\}$

3. What is the difference between the graph of $\{(x, y) \mid y < 2x - 3\}$ and the graph of $\{(x, y) \mid y \leq 2x - 3\}$? Draw the graph of each. How do you think you can show the differences?

3·12 Slope

Rundle Mountain, Banff, Alberta, has one gentle slope and one steep slope. In this picture, the right slope of the mountain is less steep than the left slope. That is, the amount of rise per unit of horizontal movement is less on the right side of the mountain than on the left side.

An engineer refers to steepness of a slope of a road by speaking of the . Gradient is the ratio of the rise of the grade to the horizontal distance of the grade. In Figure 3-14

If the road rises 6 feet every 100 feet of horizontal change, the gradient is $\frac{6}{100}$.

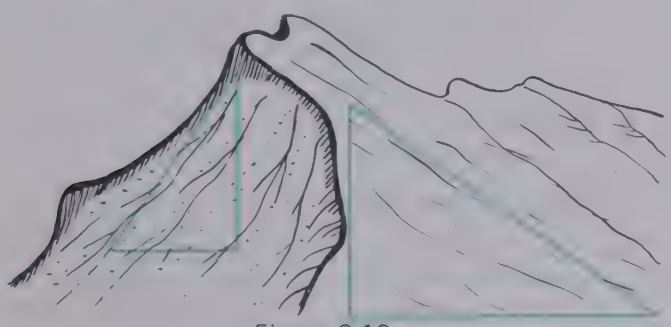


Figure 3-13

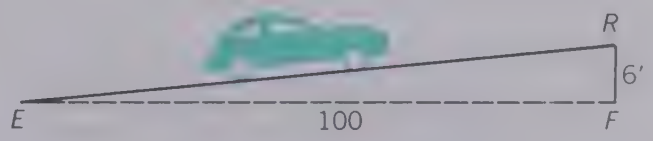


Figure 3-14

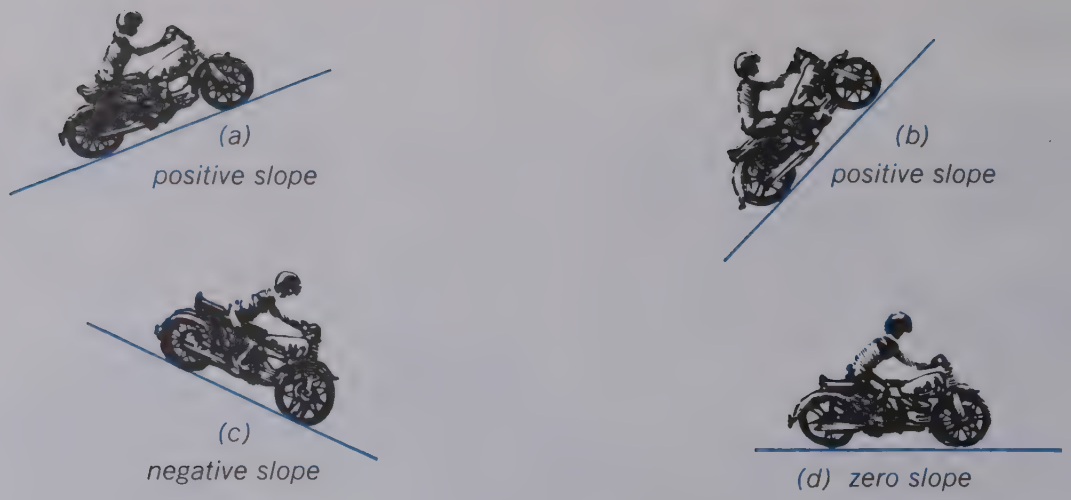


Figure 3-15

Discuss each slope illustrated in Figure 3-15. Is the slope of each steep? gentle? What do you mean by steep or gentle? Does each slope rise rapidly or slowly? What do you mean by rising rapidly or slowly? Note the difference between the positive slope and the negative slope.

In geometry, we speak of steepness of the graph of a linear function or the steepness of the line which represents the graph. To learn about slope in mathematics do the following exercises. Be very careful in answering question 2.

EXERCISE 3·12

1. Draw the graphs of the relations defined by the following equations.
   (a) $y = \frac{1}{4}x + 2$ (b) $y = x + 2$
   (c) $y = 2x + 2$ (d) $y = -\frac{1}{4}x + 2$
   (e) $y = -x + 2$ (f) $y = -2x + 2$

2. Refer to the graphs of question 1. Notice each linear equation is written in the $y = mx + b$ form.
   (a) Look at graphs (a), (b) and (c). Do the graphs rise or fall to the right? Are the values of m in the equations positive or negative?
   (b) Look at the graphs (d), (e) and (f). Do the lines (the graphs) rise or fall to the right? Are the values m in the equations positive or negative?
   (c) When m is positive the graph (rises, falls) to the right. When m is negative the graph (rises, falls) to the right.
   (d) Study the six graphs in reference to the steepness of the slopes. Which graphs have the steepest slopes? What is m for the steepest graphs?
   (e) Which graphs have the most gentle slope? What is m for these graphs?
   (f) When m is a number close to 0 (like $1, \frac{1}{2}, -\frac{1}{4}, -\frac{1}{2}$) the slope of the graph is (gentle, steep); when m is a number farther from 0 (like 2, 5, −4, −7) the graph is more (gentle, steep).
   (g) Make a rule for the steepness of a graph and the value of m.
   (h) On graph (c) label the points (−1,0), (2,6), (2,0) with the letters M, N, P, respectively. In another color, draw $\overline{MP}$ and $\overline{NP}$. What is the directed distance $\overline{MP}$? What is the directed distance $\overline{PN}$? $\overline{PN}$ is called the *rise* or *vertical*

change and $\overline{MP}$ is called the *run* or *horizontal change*. Label each segment with the appropriate name.

(i) We said earlier that steepness of a road is measured by the ratio of the rise of the road to the horizontal change. In geometry we measure slope of a graph in a similar way.

> Slope is the ratio of the directed distance of the rise to the directed distance of the horizontal change.

In graph (c), what is the directed distance of the rise $\overline{NP}$? This distance is the vertical change.

(j) What is the directed distance of the horizontal change?

(k) Slope can be written this way:

$$\frac{\textit{directed distance of the rise}}{\textit{directed distance of the horizontal change}}$$

or

$$\frac{\textit{vertical change}}{\textit{horizontal change}}.$$

Let us use the symbol $(\overline{PN})$ for directed distance of the segment $\overline{PN}$. What symbol would you use for the expression "directed distance of the segment $\overline{MP}$"?

(l) Complete the statement for graph (c).

$$\text{Slope of } \overline{MN} = \frac{(\overline{PN})}{(\overline{MP})}$$
$$= \blacksquare$$
$$= 2.$$

(m) On graph (f), label the points $(2,-2), (-2,-2)$ and $(-2,6)$ with the letters R, T, S, respectively. Draw $\overline{RT}$ and $\overline{TS}$.

(n) What is $(\overline{RT})$? What is $(\overline{TS})$?

(o) What is the slope of $\overline{RS}$?

(p) How are the measure of slope and the value of m related?

(q) Look at equation (b). What is the slope of graph (b)? Check by using points similar to the method used above.

(r) What is the slope of graphs (a), (d), (e)?

3. Consider five lines whose slopes are 5, -4, $\frac{9}{2}$, $-\frac{1}{2}$, $\frac{7}{8}$. What is the slope of the steepest line? the most gentle slope?

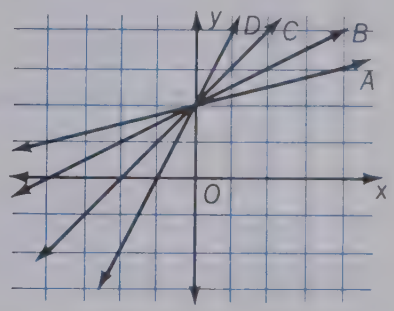
4. The four lines in the figure have slopes of $\frac{5}{4}$, $\frac{5}{8}$, $\frac{1}{4}$, $\frac{18}{8}$, but not necessarily in that order. Without measuring, give the slope of each line.

Slope of A =

Slope of B =

Slope of C =

Slope of D =



5. The slopes of the five lines are -7, 1, -1, $\frac{1}{4}$, 3. Without measuring, give the slope of each line.

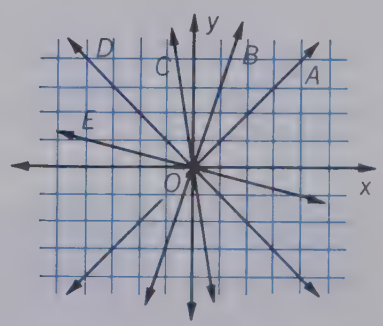
Slope of A =

Slope of B =

Slope of C =

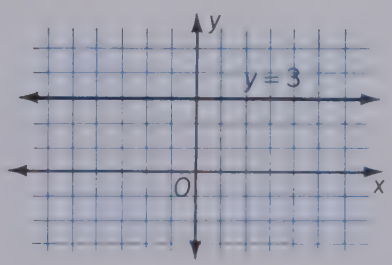
Slope of D =

Slope of E =



6. On a Cartesian plane draw lines with slopes of $\frac{1}{2}$, 1, 2, -2, -3, 4. Label them A, B, C, D, E, F, respectively.

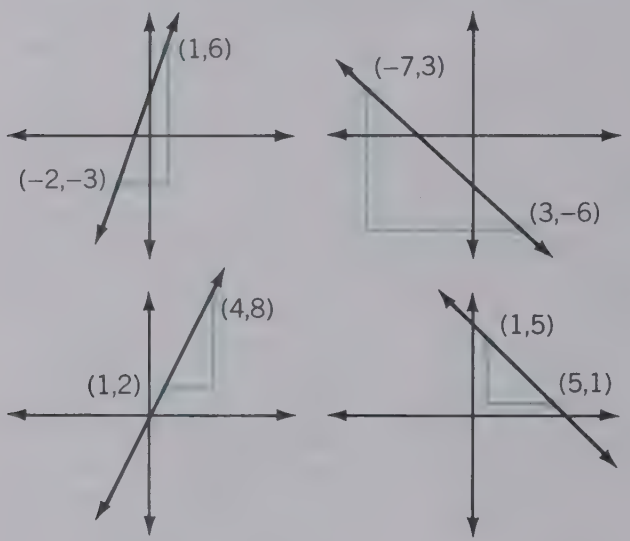
7. What is the slope of the line $y = 3$? Write $y = 3$ in the $y = mx + b$ form.



8. Consider the function $\{(x,y)|y = mx + b\}$.
   (a) For what value of m does the graph of the function have a slope of 2? -3? $-\frac{5}{2}$?
   (b) For what value of b does the graph of the function have a y-intercept of -2? 8? 0?

9. Write the functions in the form $\{(x,y)|y = mx + b\}$:
   (a) slope is 3; y-intercept is 2.
   (b) slope is $\frac{1}{4}$; y-intercept is 10.
   (c) slope is 0; y-intercept is 0.
   (d) slope is 1; y-intercept is 1.

10. Find the slope of each graph shown.



11. Find the slope of each line that contains the points:
   (a) (4,3), (8,5)
   (b) (10,4), (−2,−2)
   (c) (9,5), (4,5)
   (d) (2,−3), (−1,−1)
   (e) (0,0), (4,−1)
   (f) (7,4), (14,8)
   (g) (9,8), (9 + 2, 8 + 5).

12. We can use the "y-intercept, slope" method of graphing more easily now. Let us graph
$$\{(x,y)|y = -3x + 4\}.$$
Steps:
   1. The b tells you the y-intercept. Label the point where the graph crosses the y-axis W.
   2. The m tells you the slope. Does the graph rise or fall to the right? If the horizontal change is 1, what is the rise or fall?
   3. Placing your pencil on point W move 1 unit to the right, then move ___ units (up, down). Label this point V.
   4. Draw a line to contain W and V. This line is the graph of the function.

13. Using the procedure in Exercise 12 graph the following:
   (a) $\{(x,y)|y = 3x - 2\}$
   (b) $\{(x,y)|y = -5x - 4\}$
   (c) $\{(x,y)|y = -\frac{1}{2}x + 4\}$

14. When a carpenter speaks of the slope of a roof he uses a special term called *pitch*. Pitch is the ratio of the rise of the roof ($\overline{CD}$) to the span of the roof ($\overline{AB}$). Notice how this differs from the definition we have been using for slope. What is the pitch of the roof?

15. Engineers often have difficulty building roads with suitable grades. One example is the difficulty experienced in building the Canadian Pacific railroad through the Kicking Horse Pass in British Columbia. Prior to 1908, the railroad bed had such extreme grades that four auxiliary locomotives were required to haul a trainload of freight over this section. For about three miles a gradient of 4.5 percent prevailed—a grade of 10 times the maximum gradient normally allowed on prairie grades. In 1908 the Spiral Tunnels were completed and the gradient reduced to an average of 2.2 percent. Tunnel number one, 3200 feet in length, turns under Cathedral Mountain. The exit portal is 54 feet above the entrance portal. Tunnel number two, 2890 feet long, under Mount Ogden, has a difference in elevation of 45 feet between portals. Now two engines can haul a bigger load than the four previously used. The

tunnels double the railroad upon itself twice to form a rough figure 8 in shape.

(a) Gradients are sometimes expressed in percent as we see above. Express a gradient of 4.5 percent as a ratio with a horizontal change of 100 units. Express 2.2 percent in the same way.

(b) Express as a percent to the nearest tenth, the average gradient of tunnel number one; tunnel number two.

White dotted lines represent the route followed by Canadian Pacific trains through the famous Spiral Tunnels.

3·13 Intersection of Sets

Consider the sentence "John is going to a technical school and he is studying electronics." If the sentence is true we know:

1. John is going to a technical school.
2. John is studying electronics.

In order that the sentence be true, these two sentences must be true. Note, in order that the compound sentence be true, *both* parts must be true.

Consider the sentence

$$2 + 4 = 6 \quad \text{and} \quad 3 + 5 = 9$$

Is $2 + 4 = 6$? Is $3 + 5 = 9$? Are both parts of the sentence true? Is the sentence

"$2 + 4 = 6$ and $3 + 5 = 9$" true?

Consider "$3 < 9$ and $4 > 3$". Is $3 < 9$? Is $4 > 3$? Is the compound sentence "$3 < 9$ and $4 > 3$" true?

The sentence "$x + 5 = 6$ and $3 - x = 2$" is true only if the replacement for x is such that *both* $x + 5 = 6$ *and* $3 - x = 2$ are true. What replacement for x will make $x + 5 = 6$ true? What is the replacement for x which will make $3 - x = 2$ true? Then what is the replacement for x which makes the sentence "$x + 5 = 6$ and $3 - x = 2$' true? The replacement for x is 1. This is the solution for the compound condition.

A compound condition consisting of two equations is called a *system of equations*.

EXERCISE 3·13

1. Which of the following compound sentences are true?

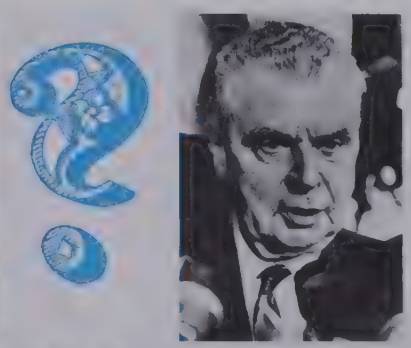
   (a) Mr. Diefenbaker was Prime Minister of Canada and Mr. Trudeau is Prime Minister of Canada.

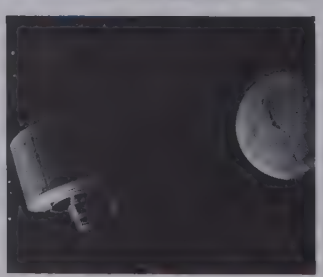
   (b) Nancy Greene won the Olympic Ski Championship and Nancy Green retired from Olympic competition.

   (c) British Columbia is the most westerly province in Canada and Nova Scotia has the most southerly point in Canada.

   (d) Newfoundland was the last province to join Confederation and Newfoundland joined the Dominion in 1949.

   (e) Canada was the first to put a satellite into orbit about the earth and Russia launched Sputnik.





3·14 Graphic Solution of a System of Equations

The following example illustrates a method to solve a system of equations. Answer the questions in the solution.

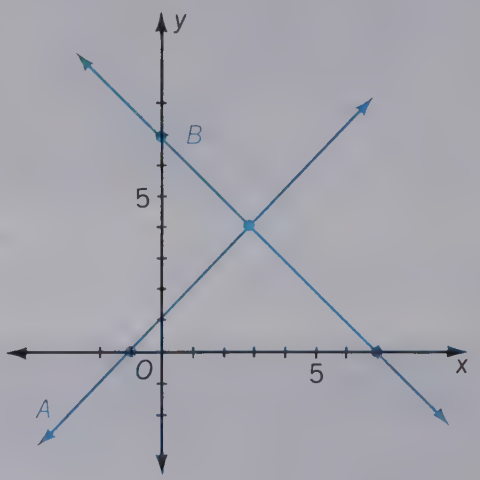
Example 1

Find the solution set of the system of equations

$$\begin{cases} (A)\ y = x + 1 \\ (B)\ y = -x + 7 \end{cases}$$

Solution

Draw with reference to the same set of axes the graph of the relation defined by each equation.



(a) Is the point (1,2) in line *A*? Name some points in line *A*. Every point in line *A* is a solution for equation *A*.

(b) Is the point (1,2) in line *B*? Name some points in line *B*. Every point in line *B* is a solution for equation *B*.

(c) Is the point (3,4) in line *A*? Is it in line *B*?

(d) Is the point (3,4) in both lines? Is the point (3,4) common to both lines? What is the point of intersection of the two lines?

(e) Does the ordered pair (3,4) make equation *A* true? equation *B*?

(f) What ordered pair makes both equations true?

(g) What is the solution of this system of equations?

$$\begin{cases} y = x + 1 \\ y = -x + 7 \end{cases}$$

(h) Verify the solution by making the appropriate replacements in A and in B.

Consider this very simple example to illustrate how we can solve problems by finding the solution to a system of equations in two variables. The example is in step form only to help you understand. Not every step is necessary in every question.

Example 2

The sum of two numbers is 10. The second number is equal to twice the first plus 4. Find the two numbers.

Solution

Step 1: Express the information given in equation form using the variable x as the replacement for the first number and y for the second number.

(1) $x + y = 10$ (first sentence)

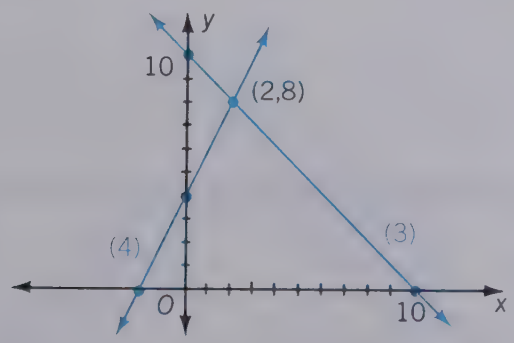
(2) $y = 2x + 4$ (second sentence)

Step 2: Express each equation in y-form.

(3) $y = -x + 10$

(4) $y = 2x + 4$

Step 3: Graph the relation defined by each equation.



Step 4: Determine the point of intersection of the two graphs. This will tell us the x-value and the y-value that satisfy both equations.

The coordinates of the point of intersection are (2,8).

Step 5: Translate the information into an answer to the question. The ordered pair (2,8) tells us the replacement for x is 2 and the replacement for y is 8. Hence the first number is 2 and the second number is 8. (This is the answer).

Step 6: Verify: Refer to the equations in step 1.

(1) $2 + 8 = 10$

(2) $8 = 2 \times 2 + 4$

EXERCISE 3·14

1. Complete this question.

Solve the system of equations

$$\begin{cases} 3x + 2y = 22 \\ y - 4x = -11 \end{cases}$$

Solution

(This time we start with Step 2 in the above example).

Step 2: Express in y-form

(1) $y = -\frac{3}{2}x + 11$

(2) $y = 4x - 11$

Step 3: Draw the graph of the relation defined by each equation

Step 4: Determine from the graph the point of intersection of the two graphs. The coordinates of the point of intersection are (?, ?).

Step 5: The solution set is (?, ?).

Step 6: Verify:

(1) $3(?) + 2(?) = 22$

(2) $(?) - 4(?) = -11$

Write a flow chart for this solution.

2. Solve the following systems of equations by the graphical method. Verify each.

(a) $\begin{cases} y = x - 3 \\ y = 2x - 10 \end{cases}$ (b) $\begin{cases} y = -5 \\ y = -x \end{cases}$

(c) $\begin{cases} y = 2x + 10 \\ y = -2x - 10 \end{cases}$ (d) $\begin{cases} y = -2x + 2 \\ 2y = x + 14 \end{cases}$

(e) $\begin{cases} y = 4 \\ 2y + 5x = 0 \end{cases}$ (f) $\begin{cases} 3y - 2x = 4 \\ 4y - x = -1 \end{cases}$

3. This question is worked the same as the above questions but the answer may surprise you. Complete the solution carefully and answer the questions.

(a) Solve the system of equations

$$\begin{cases} y = x + 4 \\ y = x - 4 \end{cases}$$

Solution

Step 3: Draw the graph of the relation defined by each equation.

Step 4: (i) Do the lines appear to intersect? Is there a solution?

(ii) What is the slope of the first graph? What is the slope of the second graph?

(iii) What is the y-intercept of the first graph? of the second graph?

(iv) Look at the two graphs. How would you describe the relationship of the two lines?

Step 5: The solution set is _____ .

(b) Draw the graphs of the relations defined by each system of equations.

(i) $\begin{cases} y = -2x - 2 \\ y = -2x + 4 \end{cases}$

(ii) $\begin{cases} y = 3x + 6 \\ y = 3x - 0 \end{cases}$

(iii) $\begin{cases} y = -\frac{2}{3}x - 4 \\ y = -\frac{2}{3}x + 6 \end{cases}$

(iv) How is each pair of lines related?

(v) Look at the m-values in each pair of equations. How are the equations in each pair related? What do they have in common?

(vi) Is there a nonempty solution set for each system?

(vii) What is the solution set for two graphs which are parallel lines.

4. This is an important question. Put your best effort into this one. Solve the system of equations. Verify.

$$\begin{cases} y = 3x - 4 \\ y = x + \frac{1}{2} \end{cases}$$

(a) Did your answer check? Did you have difficulty in determining from the graph the correct answer? What suggestions would you make to improve the accuracy of the graphs?

(b) This solution only involved *quarter* units and it is likely most students have difficulty getting a solution which checked. Do you think graphs could be used to determine solutions such as $(\frac{1}{9}, \frac{17}{33})$ or $(-\frac{21}{31}, \frac{4}{65})$?

Are your graphs accurate enough to show these solution sets?

(c) Use graphs to solve these systems of equations and *guess* the coordinates of the intersections. Do not verify. Compare your answers with those in the key.

(i) $\begin{cases} y = -x + 4 \\ y = -3x + 7 \end{cases}$ (ii) $\begin{cases} y = 2x - 5 \\ y = -x + 11 \end{cases}$

Is the graphical method of solution satisfactory for these equations?

Solve the following problems by using equations in two variables. Use the graphical solution.

5. The length of a signboard is one foot more than the width. The length is 3 feet less than twice the width. Find the length and width of the signboard.

6. The gas tank on car Y holds 4 gallons more than the tank on car X but it holds 4 gallons less than twice that of car X. How many gallons does each tank hold?

7. The sum of two numbers is 22. The difference between the two numbers is 4. What are the two numbers?

8. Judy used twice as much ribbon to wrap her Christmas gifts as Jill did. Also Judy used 6 feet less than four times that which Jill used. How many feet of ribbon did each girl use?

3·15 Algebraic Solution of a System of Equations

Question 4 of the preceding exercise illustrated that the solution cannot always be easily nor correctly read from the graphs. It becomes evident that there is a need for some other method of solution. Consider the system of equations:

$$\begin{cases} y = -x + 8 & (1) \\ y = x + 1 & (2) \end{cases}$$

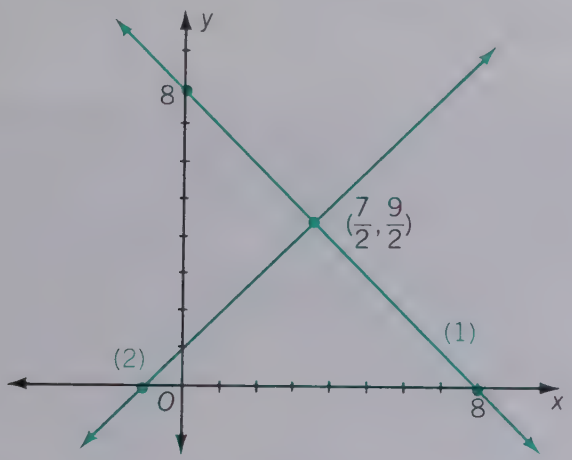


Figure 3-16

Geometrically, at the point of intersection the y-value of graph (1) is the same as the y-value of graph (2), and the x-value of graph (1) is the same as the x-value of graph (2).

Therefore, we know

1. when $x = \frac{7}{2}$, the value of y in equation (1) is the same as the value of y in equation (2).
2. when $y = \frac{9}{2}$, the value of x in equation (1) is the same as the value of x in equation (2).

Knowing that the y-values in the solution set are equal in this system of equations

$$\begin{cases} y = -x + 8 & (1) \\ y = x + 1 & (2) \end{cases}$$

we can write

then $$\left.\begin{aligned} -x + 8 &= x + 1 \\ -2x &= -7 \\ x &= \tfrac{7}{2}. \end{aligned}\right\}$$ *Write a flow chart for this solution.*

In equation (1) we replace x by $\frac{7}{2}$.

$$y = -(\tfrac{7}{2}) + 8$$
$$y = \tfrac{9}{2}$$

The solution set for the system is $(\frac{7}{2}, \frac{9}{2})$.

This method of solution is known as the *method of comparison.*

Example 1

Solve algebraically the system of equations

$$\begin{cases} y = -2x & (1) \\ y = 6x + 11 & (2) \end{cases}$$

Solution

Recalling the statement in the work above that the y-values of the two equations are equal in the solution set, we use the transitive property of equality to write

$$-2x = 6x + 11$$
$$-8x = 11$$
$$x = -\tfrac{11}{8}.$$

Making the replacement in equation (1) for x we have

$$y = -2\,(-\tfrac{11}{8})$$
$$y = \tfrac{11}{4}.$$

The solution set is $(-\frac{11}{8}, \frac{11}{4})$.

Verify:

Equation (1)

$$\tfrac{11}{4} = -2(-\tfrac{11}{8})$$
$$\tfrac{11}{4} = \tfrac{11}{4}$$

Equation (2)

$$\tfrac{11}{4} = 6(-\tfrac{11}{8}) + 11$$
$$\tfrac{11}{4} = \tfrac{11}{4}$$

Geometrically the solution for Example 1 is shown in Figure 3-17. It is evident that the solution cannot be read accurately from the graph. Therefore, the algebraic solution is necessary.

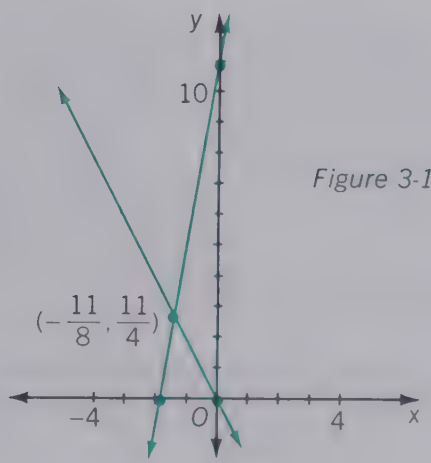


Figure 3-17

EXERCISE 3·15

1. Solve algebraically the systems of equations. Verify.

(a) $\begin{cases} y = 3x + 9 \\ y = 6x - 4 \end{cases}$ (b) $\begin{cases} y = -4x + 6 \\ y = x - 6 \end{cases}$

(c) $\begin{cases} y = 12x - 12 \\ y = x + 23 \end{cases}$ (d) $\begin{cases} y = -16 \\ y = -5x + 29 \end{cases}$

(e) $\begin{cases} y = 9x - 7 \\ y = 2x + 13 \end{cases}$ (f) $\begin{cases} y = 8x - 27 \\ y = x + 13 \end{cases}$

(g) $\begin{cases} y = -2x - 6 \\ y = -3x - 16 \end{cases}$ (h) $\begin{cases} y = 100x - 40 \\ y = 300x - 400 \end{cases}$

3·16 Problem Solving

In your previous grades you have solved problems expressed in words by means of one equation with one variable. Now that we know how to solve a system of linear equations in two variables, we will be able to do more complicated problems. Study the two examples which illustrate the algebraic solution.

Example 1

A man invests two amounts of money that total \$770. The interest on the amount he invests at 5% is equal to the interest on the amount at 6%. How much did he invest at each rate?

Solution

Let x represent the number of dollars invested at 5%.
Let y represent the number of dollars invested at 6%.
We can translate the two statements this way

$$\underbrace{\text{two amounts of money}}_{x+y} \quad \underbrace{\text{that total}}_{=} \quad \underbrace{\$770}_{\$770}$$

$$\underbrace{\text{interest on the amount invested at 5\%}}_{0.05x} \quad \underbrace{\text{is equal}}_{=} \quad \underbrace{\text{to the interest on the amount invested at 6\%.}}_{0.06y}$$

These equations are put in the proper y-form.

$x + y = 770$ is equivalent to $y = -x + 770$
$0.05x = 0.06y$ is equivalent to $y = \frac{5}{6}x$.

We then solve this system of equations by using the transitive property of equality.

$$\frac{5}{6}x = -x + 770$$
$$\frac{11}{6}x = 770$$
$$x = 420$$
$$y = 350$$

We write a conclusion:

> \$420 was invested at 5% and \$350 was invested at 6%.

We then verify: (Do these answers check?).

Some problems are worded so that the relations are readily evident and the equations can be written directly from the statements (as in the example above). Other problems are not so clearly stated. The student must interpret the information and realize the relations. The following example is the same problem as in the previous example. Study the approach used.

Example 2

How should \$770 be invested so that the income from the amount invested at 5% is the same as the income from the amount invested at 6%?

Solution

Let us for a moment pretend that you do not see the relations clearly enough to write equations.

One approach is this:

Make a guess that \$400 is invested at 5% and the balance

(1) $\$770 - \$400 = \$370$

at 6%. The second bit of information tells us that the interest (income) on each part is the same.

Hence, the interest on \$400 = interest on \$370
or (2) $0.05 \times 400 = 0.06 \times 370$
$20 = 22.20$

which obviously is not correct. Therefore we know the two amounts are not \$400 and \$370. But we have been able to analyse the problem. Now we can become more formal in our approach. If x represents

the number of dollars invested at 5% and y the number of dollars invested at 6%. We know from (1) above

$$770 - x = y$$

and from (2) above

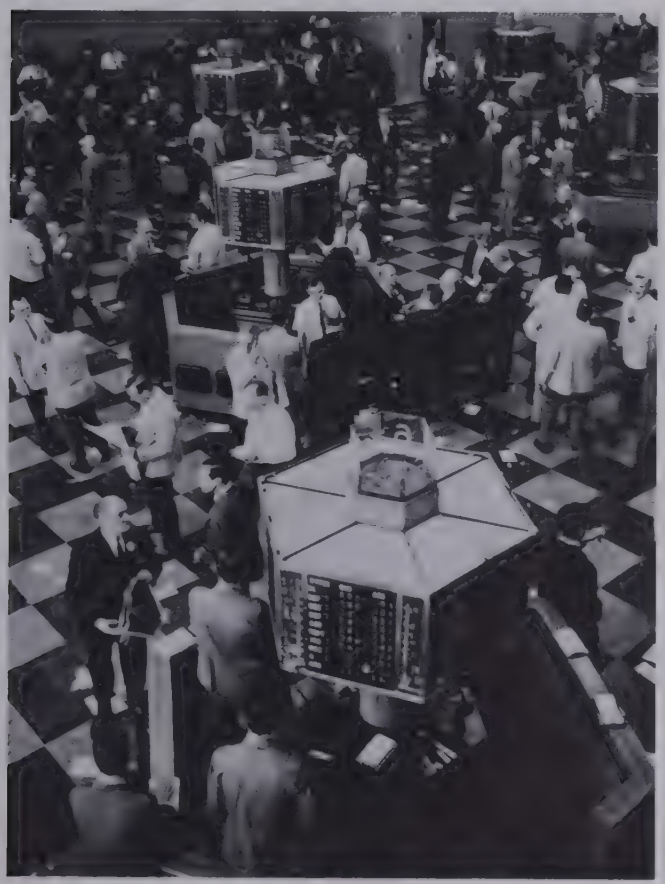
$$0.05x = 0.06y$$

We are now in a position to write these equations in the y-form and solve. We get the system:

$$\begin{cases} y = -x + 770 \\ y = \frac{5}{6}x \,. \end{cases}$$

The solution set is $\{(420, 350)\}$.

Notice how we are able to define the relations by making a guess what the answers were. Let us analyse another problem in the same manner.

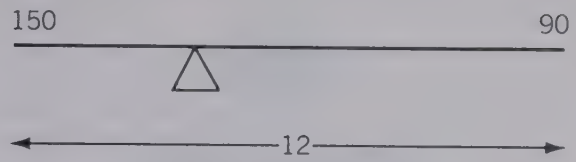


Example 3

A lever 12 feet long is on a fulcrum similar to a teeter-totter (first class lever). A boy, 90 pounds, sits on one end and a boy, 150 pounds, sits on the other end. How far from the fulcrum is each boy?

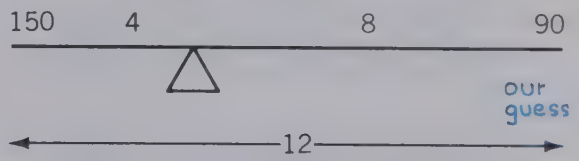
Solution

A diagram may help do this problem.



Now let us make a guess. Let the distance the 90 pound boy is from the fulcrum be 8 feet. The 150 pound boy must be

(1) $12 - 8 = 4$ feet from the fulcrum.



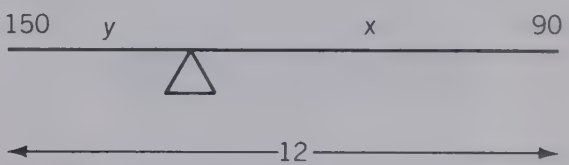
Now the fact you must know (from science) is

(2) $150 \times 4 = 90 \times 8$

$600 = 720.$

Obviously, this is not true but again it has given us insight into how this question is done.

Since our guess was wrong, let the distance the 90 pound boy is from the fulcrum be represented by x and the distance the 150 pound boy is from the fulcrum be represented by y.



Then from (1) above we know

$12 - x = y$

and from (2) we know

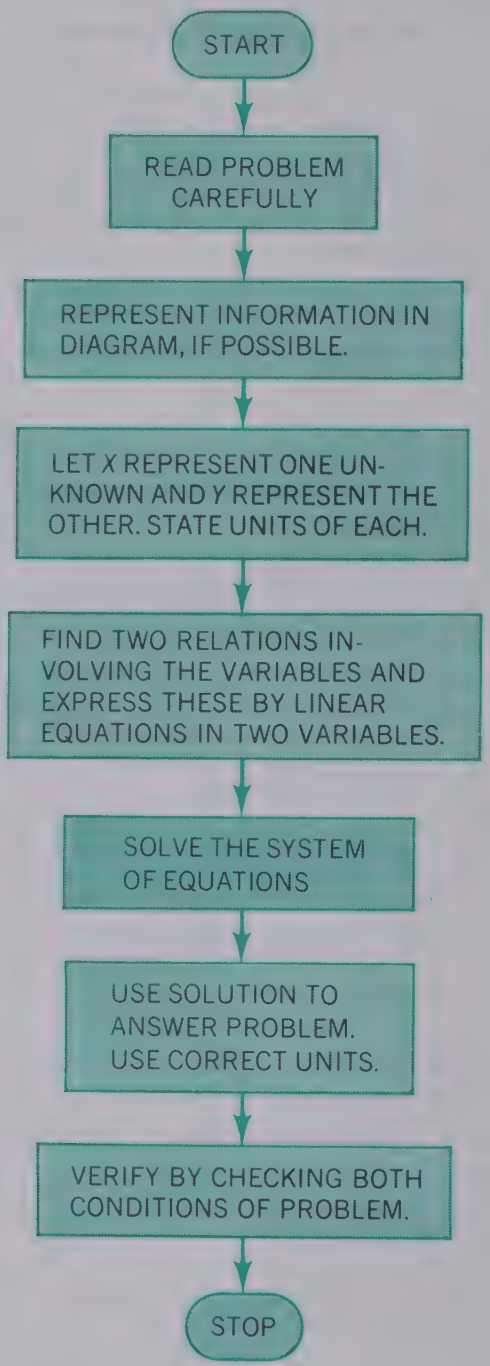
$150y = 90x$

We now proceed as in the other example. Finish this example.

Write equations in y-form.
Solve for x.
Substitute and solve for y.
State your conclusion.
Verify.

Did you find that the 150 pound boy is 4.5 feet from the fulcrum and the 90 pound boy is 7.5 feet from the fulcrum?

Recommended flow chart for solving word problems.


START
READ PROBLEM CAREFULLY
REPRESENT INFORMATION IN DIAGRAM, IF POSSIBLE.
LET X REPRESENT ONE UNKNOWN AND Y REPRESENT THE OTHER. STATE UNITS OF EACH.
FIND TWO RELATIONS INVOLVING THE VARIABLES AND EXPRESS THESE BY LINEAR EQUATIONS IN TWO VARIABLES.
SOLVE THE SYSTEM OF EQUATIONS
USE SOLUTION TO ANSWER PROBLEM. USE CORRECT UNITS.
VERIFY BY CHECKING BOTH CONDITIONS OF PROBLEM.
STOP

EXERCISE 3·17

Complete the following statements.

1. If the sum of x dollars is invested at 7.5% per annum, the annual interest is ____.

2. If premium gas is 52¢ a gallon and regular gas is 43¢ a gallon, the cost of x gallons of premium gas is ____; the cost of y gallons of regular gas ____; the cost of the $x + y$ gallons of gas is ____.

3. A rectangle has a width of x inches and a length of y inches. The perimeter of the rectangle is (in terms of x and y) ____.

4. An airplane with a speed in still air of 325 m.p.h. flies into a wind of 65 m.p.h. The actual speed of the airplane in the wind is ____. The plane flies into the wind for 4 hours. It goes ____ miles. The same plane flies with the same wind. Its actual speed is ____ m.p.h. It goes ____ miles in 6 hours. The total distance the plane travelled in the 10 hours is ____. Its average speed over the 10 hours is ____.

5. Two machines fill cartons of milk. The older machine fills x cartons per hour while the newer machine fills y cartons per hour. The number of cartons filled by both machines in 4 hours is____.

6. A man works regular hours and overtime. He receives x dollars per hour for regular time and y dollars per hour overtime. If he works 35 hours regular time and 12 hours overtime he receives ____ dollars.

For each of the following write the two equations in two variables that you would use to solve the problem. Do not solve the equations. Make statements telling what each variable represents.

7. John has \$175 in five and ten dollar bills. He has 3 times as many five dollar bills as ten dollar bills. How many of each kind of bills has John?

8. The larger of two complementary angles has a measure of 20 degrees more than three times the smaller angle. What is the measure of each angle? (Two angles are complementary if the sum of their measures in degrees is 90.)

9. A dress pattern calls for some wool fabric and some *facing* material. The cost of the wool material is \$2.80 per yard and the cost of the facing is \$1.40 per yard. The total cost of all the material is \$8.40. How much of each kind of material is required?

10. Telephone rates are based on the cost of the first 3 minutes and an additional charge for each minute over the 3 minutes. A business man phones twice from Calgary to Toronto, person to person. The first call is for 6 minutes at a cost of \$7.05. His second call is for 10 minutes and costs \$9.65. What is the charge for the 3 minutes and for each additional minute?

11. If $\frac{2}{5}$ of one of two supplementary angles is $\frac{1}{2}$ of the other, what is the measure of each of the angles? (Two angles are supplementary if their sum is 180°.)

12. The sum of the diameters of two circles is 25 inches. The circumference of the larger circle is 9.42 inches more than the circumference of the smaller circle. How long is each diameter?

13. The price of a dress and matching shoes is \$40. The dress is $1\frac{1}{2}$ times as much as the shoes. What is the price of each item?

14. Two airplanes flying towards each other from points which are 900 miles apart meet each other in $1\frac{1}{2}$ hours. There is a wind of 30 miles per hour. In still air plane *A* files 100 miles per hour faster than plane *B*. Plane *A* flies into the wind and plane *B* flies with the wind. What is the speed of each plane in still air?

Use systems of equations and algebraic solutions to solve the following problems. Verify.

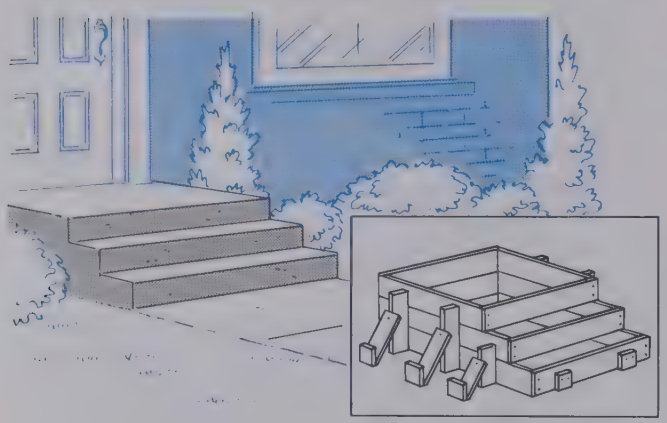
15. A man invests \$1200. Part was invested at 6% and part at 10%. The interest on the part invested at 10% was \$24 more than that on the part invested at 6%. Find the amounts invested at each rate.

16. How should \$2000 be invested so that the interest from the amount invested at 7% is \$8 more than that from the amount invested at 9%?

17. The first class lever is balanced when the heavier weight (*A*) is 4 feet from the fulcrum and the lighter weight (*B*) is 6 feet from the fulcrum. If the lighter weight (*B*) is increased by 50 pounds, the lever now balances when the fulcrum is 6 feet from *A* and 4 feet from *B*. Find the two weights.

18. An isosceles triangle has a perimeter of 228 inches. The sum of the equal sides is 100 inches more than the third side. Find the length of each side of the triangle.

19. A contractor and subcontractor receive $2,800 for a job. It was agreed that the contractor would receive $200 more than twice that which the subcontractor received. (The contractor did more work and had the initial expense involved in getting the job.) How much did each party get?



20. A gasoline company wishes to mix some gasoline at 40¢ a gallon with some at 50¢ a gallon to produce 2000 gallons at 42.3¢ a gallon. How many gallons of each kind should he use?

21. Concrete is made of 1 part cement and 5 parts sand and stone combined. A builder wants 660 yards of concrete. How much of each is required?

Using the Library

1. *Wonderful World of Mathematics* Read page 51. Explain how the early Hindu and Moslem mathematicians might have done the simple algebra question:

$$\frac{3n}{5} = 60$$

Read pages 58-59. Who first presented the theory that the earth and planets circled the sun?

2. *Life Science Library Mathematics* Read pages 80 through 85. Write a paragraph about Descartes. What is the English difference between the two stamps on page 85?

REVIEW EXERCISE 3

1. Given A = {January, February, March, . . . , November, December} and $B = \{28, 29, 30, 31\}$. Set A contains the months of the year, and the elements of set B are the number of days in a month.

   (a) Tabulate the truth set for the relation *days has*. (31 *days has* January.)

   (b) Is this relation a function? If not, tell why.

2. Transform $3x + 9y = 6$ into the form $y = mx + 6$.

3. Draw a graph of $\{(x,y)|y \geq 3x + 6\}$. Shade the appropriate half-plane.

4. Solve graphically the following system of equations:

$$\begin{cases} y = x + 10 \\ y = -2x - 2. \end{cases}$$

5. Solve algebraically the following system of equations:

$$\begin{cases} y = x + 2 \\ y = -3x + 12. \end{cases}$$

6. Write the two linear equations in two variables that you would use to solve the following problem. Identify the variables you use.

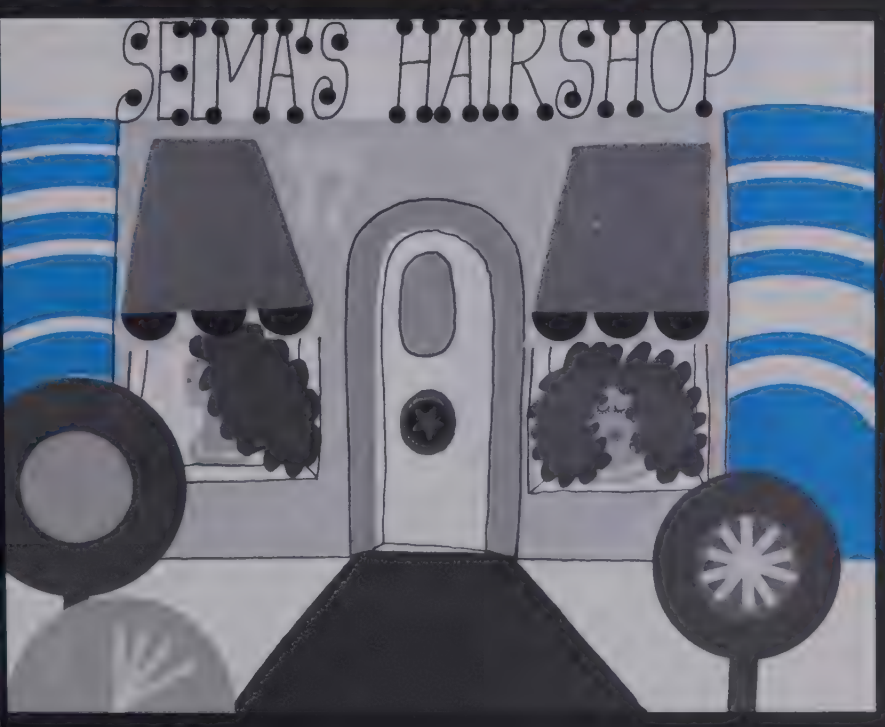
   A 170 acre piece of town land was valued at $175,000. The hill-land was valued at $700 an acre and the level-land at $1,200 an acre. How many acres of each kind of land were there?

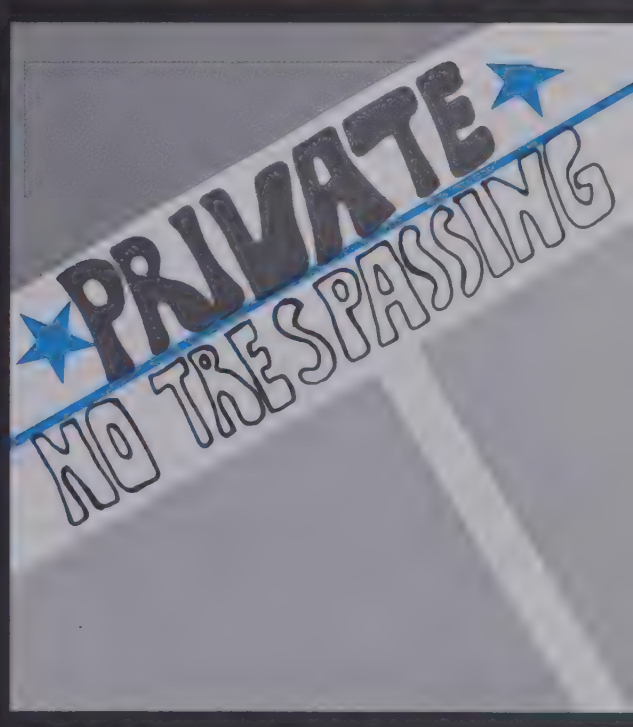
7. What is the slope and the y-intercept of this equation:

$$y = \tfrac{3}{2}x - 4.$$

Chapter Four

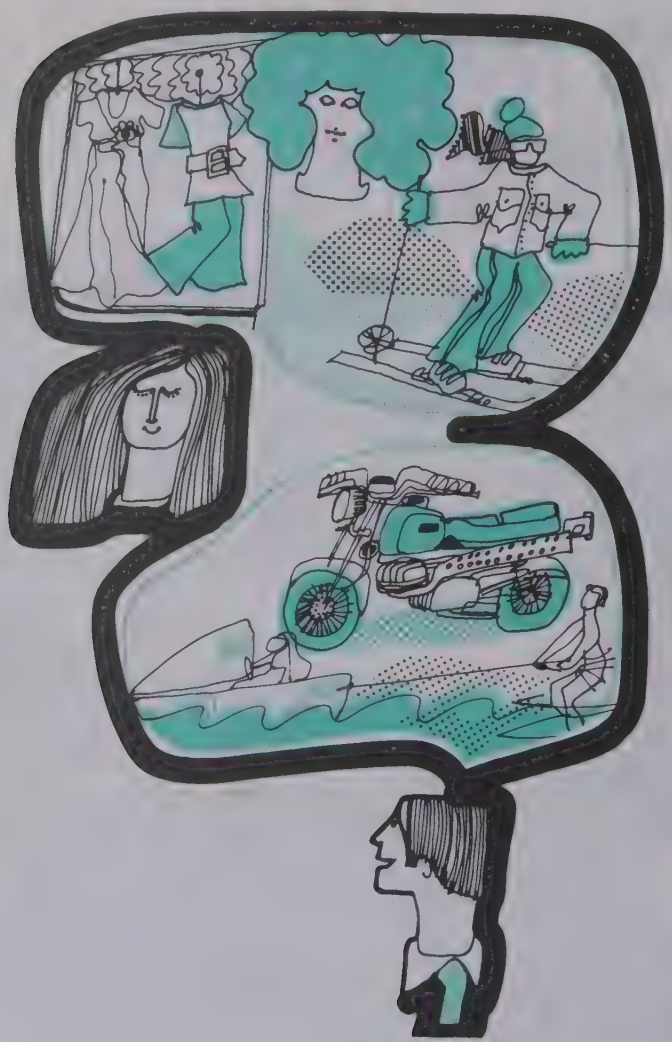
PURCHASING AND MANAGING PERSONAL PROPERTY





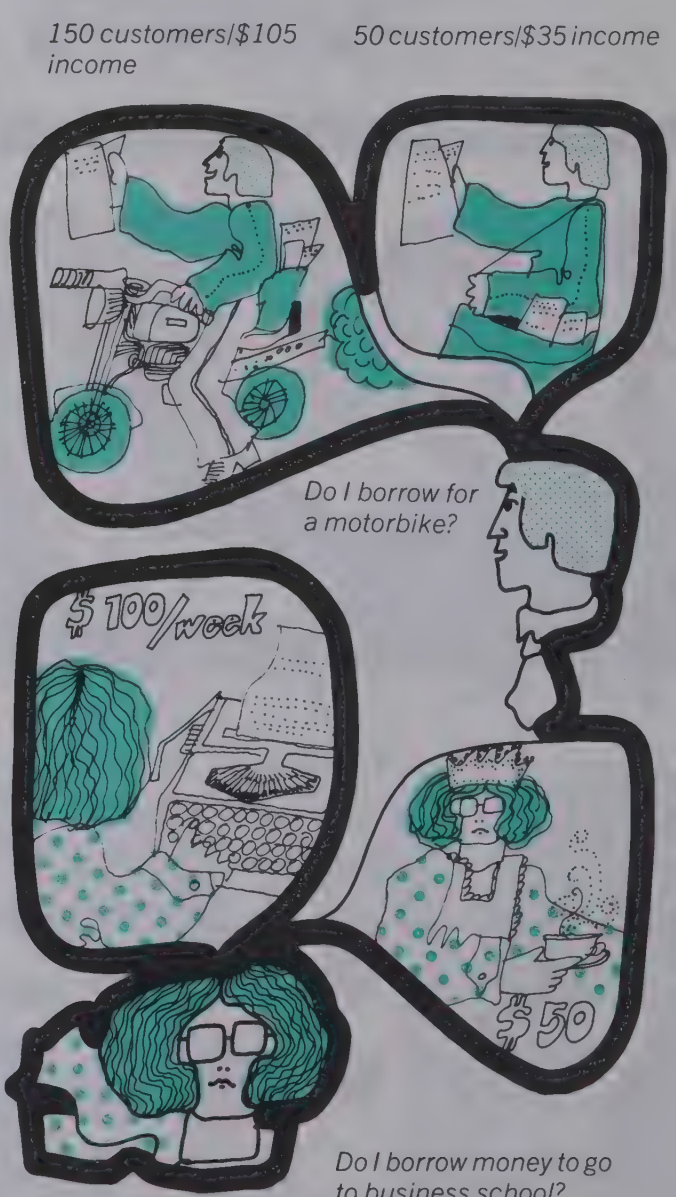
4·1 Introduction

Think of some major item you would like to buy—a motorbike, a ski outfit, a new wardrobe, a wig or a set of water skis.



But how would you pay for these? And how can you keep up the expenses that accompany some of these things you want? Do you have enough readily available money (usually referred to as)? If you do not have enough available money, but have a steady income which provides enough for an extra expenditure per pay period, you may decide to make a *down* and a *time payment* until the sum has been paid. You may decide to borrow the required money from some source such as a bank, a credit union, or a finance company. When you purchase on credit (promising to pay in the future for what you get now) or borrow money, you are using someone's money. You are expected to pay interest for the use of this money. Hence, credit buying costs more than purchasing by cash.

One should consider the advantages and disadvantages of credit buying before signing a credit contract. Unless the advantages outweight the disadvantages credit buying should not be entered into.



EXERCISE 4·1

For discussion: Divide the class into groups of about 6 to discuss each topic. Each group should have a recorder to make a list of important points. A summary of each group discussion should be placed on the blackboard for general class discussion.

1. Discuss the advantages and disadvantages of credit buying.
2. Discuss the advantages and disadvantages of cash buying.
3. For what good and valid reasons and under what circumstances might
   (a) a girl buy a wardrobe on credit?
   (b) a boy buy a motorbike (or car) on credit?
   (c) a high school graduate borrow money to go to a post high school institution for further education or training.

4·2 Credit Buying

We will consider the difference in cost of cash buying and credit buying. While there are a number of ways of buying on credit they come under two headings: *single payment* and *instalment payments*. We first examine single payment.

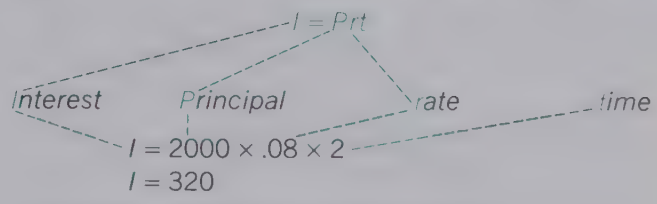
Example 1

A young fisherman purchases a fishing boat for $2400. He borrows $2000 from the bank and takes $400 from his savings account. He agrees to pay the bank the $2000 two years from the date of purchase. The bank requires 8% per annum simple interest. Determine the total amount the boat cost the fisherman.

Solution

The cost of the boat is
cost price + interest on borrowed money

Let us find the interest on $2000 for 2 years at 8%



$I = 2000 \times .08 \times 2$

$I = 320$

The total cost is $2400 + $320 = $2720.

Example 2

A customer borrows $350 for 90 days at 9%. (Rate is per annum unless otherwise stated.) Calculate the interest he must pay and the amount he must pay.

Solution

Calculate the interest on $350 for 90 days at 9%.

$I = Prt$

$I = 350 \times .09 \times \frac{90}{365}$

Note the time is expressed in years!

$I = 7.77$

The interest is $7.77.

The amount is the principal + interest:

Amount = 350 + 7.77

= 357.77

The amount to be repaid is $357.77

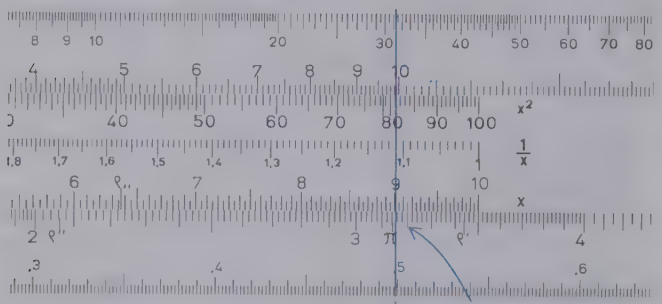
The slide rule can be used to make computations easier if calculators are not available. Note how a slide rule can be used to make the calculations required in Example 2.

$$350 \times .09 \times \frac{90}{365}$$

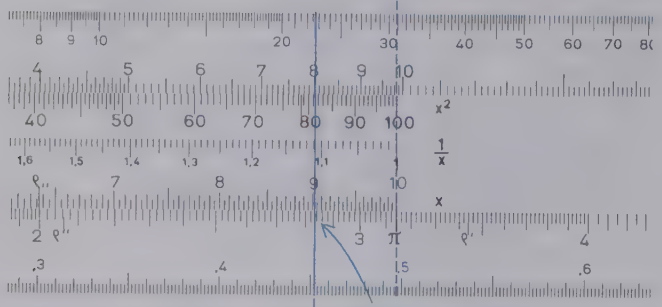
Approximate: $4 \times 10^2 \times 10^{-1} \times 10^2 \times \frac{1}{4 \times 10^2} \simeq 10$

Use: $$\frac{3.5 \times 10^2 \times 9 \times 10^{-1} \times 9 \times 10}{3.65 \times 10^2}$$

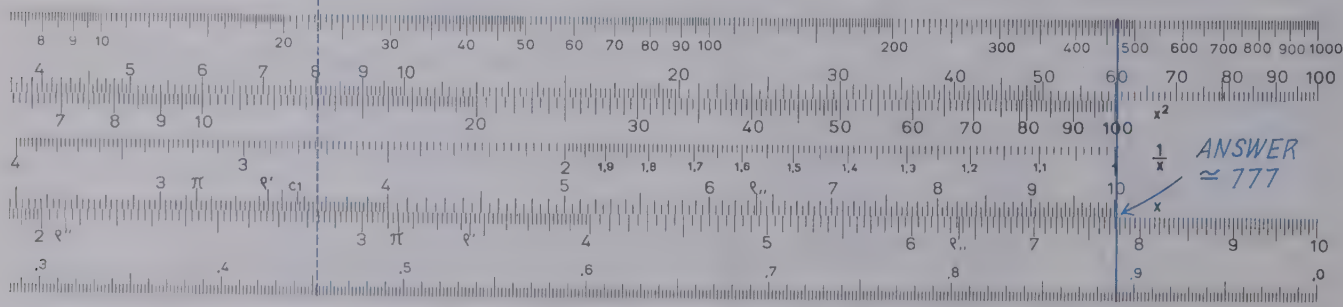
Multiply 3.5×9. Place hairline over answer.



Multiply by 9 again.
$3.5 \times 9 \times 9$



Divide by 3.65
$$\frac{3.5 \times 9 \times 9}{3.65}$$



The digits and approximation tell us the answer is $7.77.

EXERCISE 4·2

1. Find the amount of interest on a bank loan of $356 for 2 years at 7% interest.

2. Mary Jane who has an office job, bought a wig for $50. She paid $8 down and agreed to pay the balance two months later. The interest was 12% per year. Find the cost of the wig.

3. In order to buy a pair of water skis Mark had to sign a contract agreeing to pay in 2 months' time $70 plus interest at 18% per annum. How much interest did he pay?

4. Find the amount of interest on each loan:

	Principal	*Interest rate*	*Time*
(a)	$240	12%	2 years
(b)	$78	$9\frac{1}{2}$%	3 months
(c)	$182	$10\frac{1}{4}$%	150 days
(d)	$250	$7\frac{1}{2}$%	300 days
(e)	$580	$6\frac{1}{4}$%	5 months

5. George paid $72 interest on a loan of $400 for 2 years. What interest rate was he charged?

6. What interest rate was charged if $97.50 interest was charged on a loan of $500 for 3 years.

7. Henry traded his old bike in on a new motorbike. He was told the interest at 11% on the payment to be made in 2 years' time would be $55. What was the principal?

Use a calculator or slide rule wherever appropriate in this Chapter.

8. The interest on a 2 year loan is $6.75. The rate is 18%. What is the principal?

9. Given $I = Prt$ solve for
   (a) r (b) t (c) P

4·3 Credit Buying Continued

The person who wishes to buy personal property from a store but does not wish to pay cash will be faced with the problem of choosing from several plans to use. Some plans are listed here with a brief description of each. Not every store or business has every plan. Usually, a store will have one or more available to its customers.

A. Deposit Account

1. A sum of money is deposited with the business firm. Periodically more money is deposited.
2. Customer charges to his account the item he buys.
3. There should be a credit balance at all times.
4. Interest is paid by the firm on the credit balance.

B. Charge Accounts

1. Customer must open an account with the business firm. He signs an agreement.
2. Customer merely charges the items he purchases by presenting a *credit card* and signing a sales slip.

3. Once a month on a stated date, a statement of the customer's account is mailed to him.
4. It is expected the account will be paid in full within a certain period of time. If it is there is no interest charge.
5. If the account is not paid in full within the specified time a service charge is added to the next month's statement.
6. There is a maximum of credit granted. Hence this account is used for small purchases.
7. Most oil company credit cards are of this type.

C. Revolving Credit Plan

1. Customer establishes his credit by
   (a) filling in a contract giving his financial and employment record.
   (b) listing business and personal references.
2. Customer receives a credit card.
3. Customer charges items as with a charge account.
4. There is a maximum of credit granted.
5. The firm mails a statement showing previous month's balance, service charges, purchases and minimum payment required that month.
6. If the account is paid in full, there is no service charge. If a balance remains a service charge is made based on the unpaid balance.

D. Deferred Payment

1. Usually called *Instalment Plans* or *Time Payment Plans*.
2. Usually used for large purchases.
3. Customer signs a *Conditional Sales Agreement* which sets out the amounts and dates of payments.
4. Usually contract states that if a payment is in default (not paid on time) all remaining instalments shall become due and payable without notice. The company may seize the goods if payment is not made.
5. Often a down payment or trade-in is made. The balance plus sales tax (if any) and *carrying charges* (mostly interest) are calculated. This total is divided by the number of payments to obtain the amount each payment will be.

E. All Purpose Accounts

1. A combination of Charge Account and a Deferred Payment Account.
2. Usually an All Purpose Account Card is issued upon receipt of an application – if credit rating is approved.

3. Purchases are made by presentation of card and signing the sales slip.
4. This account allows for purchases of larger items without the need to sign a *Conditional Sales Agreement*.
5. Firm issues monthly statements showing previous month's balance, carrying charges, purchases and minimum payment.
6. Some oil company credit cards are of this type.

F. General Charge Cards – (American Express, Carte Blanche, etc.)

1. These cards are honoured by many different companies in many countries.
2. Some of these cards must be purchased for a small annual amount.
3. Purchases are made by presentation of the card.
4. Accounts are payable upon receipt of statement.

G. Bank Cards, ChargeX, etc.

1. Credit cards are obtained from the bank where your account is.
2. Purchases are made by presentation of card and sales slip.
3. Sales slip is sent to bank and payment is made by the bank and debited to your account.
4. Record is presented to you by your bank.
5. If your purchases are more than your bank account, interest is charged on a loan granted automatically.

EXERCISE 4·3

1. Invite the credit manager of a department store to talk with your class. Outline those areas you would like to discuss.
   Some topics are listed here:
   Types of accounts – explanation of each.
   Advantages and Disadvantages of credit buying.
   Which type of account is the most expensive to the customer.
   Which account is the easiest to open.
   Limits of accounts.

2. List those things that are common to all accounts listed in this section.

3. Which account makes it the easiest for the user to overspend? the hardest to overspend?

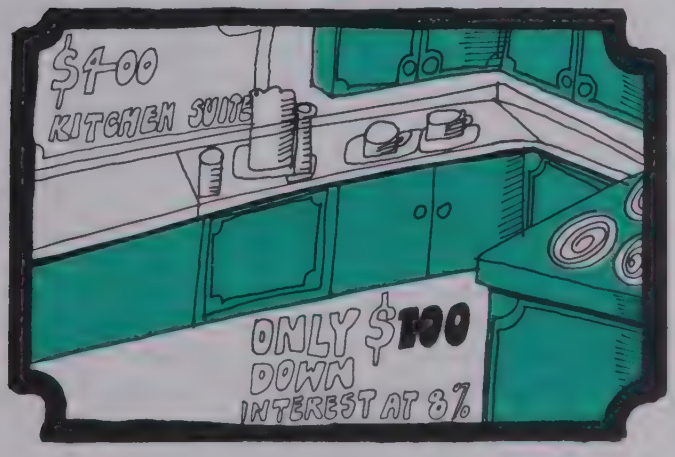
4. What are the advantages to the firm in issuing credit cards?

5. Obtain from mail order catalogues the *Payment Account Agreement.*
   (a) What information is required on this application?
   (b) What does it say about termination of the agreement?
   (c) What does it say about default of payments?
   (d) What does it say about all purchases made in your name?

6. Obtain from an oil company an application form for a credit card. Also obtain the agreement issued with the card.
   (a) What obligations are placed on the holder of the card?
   (b) What is said about lost cards?
   (c) What is said about payments?

7. A mail order catalogue usually contains charts for minimum payments and service charges on different accounts. Obtain these charts. Why do you think there are different payments and service charges for different accounts?

8. A newly-wed couple buys a $400 kitchen suite. There is a 5% sales tax. They pay $100 down. The balance has an interest charge of 8%. There are to be 12 monthly payments. Find the amount of each deferred payment.



9. A lumberjack buys a power saw priced at $350 plus 8% sales tax. He makes a down payment of $78. He agrees to pay the balance in one year. The interest rate is 12% on the balance. What is the amount the lumberjack must pay to settle the account?

10. Consult your local library (or send to "Life" at Time Inc., 541 N. Fairbanks Court, Chicago, Illinois 60611) for "A Gift from Your Friendly Banker" in the March 27, 1970 issue of "Life". This is an *eye opener* about the responsibilities one assumes when accepting a credit card from a company. While written primarily for the United States, most of the article applies to Canada as well.

4·4 Buy On Credit

ACCOUNTS PAYABLE AT STORE

TRANSACTIONS MADE AFTER BILLING DATE WILL APPEAR ON NEXT STATEMENT

PLEASE DO NOT MAIL CASH

MISS HELEN ROSS
4545 MAIN STREET
YOURTOWN

PAYMENT ENCLOSED

BILLING DATE Oct. 27/71

PLEASE RETURN THIS PORTION WITH REMITTANCE

P. P. and R. DEPARTMENT STORE

ACCOUNT NO. 494-056-5

TRANSACTION DATE		TRANSACTION	ADDITIONS		DEDUCTIONS		NEXT BILLING DATE
28	09	Payment			75	–	NOV. 28 71
1	10	Carrying Chge.		43			NO SERVICE CHARGE IF BALANCE PAID BEFORE NEXT BILLING DATE
1	10	Purchase	23	50			
2	10	Returns			13	–	ACCOUNTS ARE SUBJECT TO A CARRYING CHARGE OF 1¾% PER MONTH (21% PER ANNUM) ON THE BALANCE UNPAID BETWEEN BILLING DATES.
15	10	Purchase	42	25			
		PREVIOUS BALANCE					NEW BALANCE / MINIMUM PAYMENT
		100	66	18	88	–	78.18 / 8.00

Miss Helen Ross received the above statement of her account.

The back of the bill is shown here.

MONTHLY PAYMENT SCHEDULE

IF BALANCE IS	MINIMUM MONTHLY PAYMENT REQUIRED
0 - 20.00	$5 or the balance if under $5
20.01 - 40.00	$ 6.00
40.01 - 60.00	$ 7.00
60.01 - 80.00	$ 8.00
80.01 - 100.00	$ 9.00
100.01 - 150.00	$12.00
150.01 - 200.00	$15.00

Do question 1 of Exercise 4·4.

Example 1

Charles charges $64.40 on his Star Oil Credit Card in January. At the end of January he receives the statement of his account. He pays $45 in February. He also charges $50.60 in February. Carrying charge rate is 1½% per month on the unpaid amount. Determine the new balance on his statement at the end of February.

Solution

January:

	Purchases	$64.40
Less:	Payment	$45.00
	Balance	$19.40
Add:	Carrying charge	$ .29 (19.40 × .015 × 1)
Add:	Purchases	$50.60
	New Balance	$70.29

The new balance is $70.29.

EXERCISE 4·4

1. Check the account Miss Ross received to make certain it is correct. (This is a good procedure to follow since mistakes are sometimes made. You do not want to be charged for someone else's purchases recorded on your account by mistake.)

2. Miss Ross had these transactions in November

Nov. 5	Payment	$50.00
Nov.10	Purchase	$39.50
Nov.15	Returned goods	$12.25
Nov.22	Purchases	$92.60

   Draft a statement of her November account. Don't forget to calculate service charge.

3. In November, Muriel purchases $86.60 worth of goods on a New All-Purpose Account. At the end of November she pays $70 on her account. In December she makes purchases worth $62.50. The agreement is that there will be a service charge of $1\frac{1}{2}\%$ per month on all unpaid accounts. What will be the amount of her account at the end of December?

4. Determine the new balance in each case:

Previous Balance	Monthly rate of Service Charge	Service Charge	Purchases	New Balance
(a) $72.70	$1\frac{1}{2}\%$	?	$47.35	?
(b) $39.50	$1\frac{3}{4}\%$	?	$27.53	?
(c) $96.00	$1\frac{1}{4}\%$	?	$42.15	?
(d) $27.42	$1\frac{1}{2}\%$	?	$63.55	?

4·5 Deferred Payments

Many people do not give much thought to deferred payments. They just accept the statements they are given. A purchaser should know how much he is paying for the money he is borrowing.

Example 1

Mr. and Mrs. Knox purchased a living room suite for $440. They made a down payment of $80 and agreed to pay the balance in 6 equal payments. Interest (service charge) is charged at the rate of 18% per annum. Calculate the monthly payments.

Solution

Cost	$440
Down Payment	$80
Balance	$360

Interest on $360 at 18% for 6 months is:

$$I = Prt$$
$$= 360 \times .18 \times \tfrac{1}{2}$$
$$= 32.40$$

Total to be paid:

Balance	$360
Interest	32.40
	$392.40

Amount of each payment $\frac{392.40}{6} = 65.40$

Each payment is $65.40.

EXERCISE 4·5

1. Henry purchased the materials to complete his dune buggy for $525 from an auto supply store. He paid $75 down and agreed to pay the balance in 8 monthly payments. The service charge rate was $1\frac{1}{2}\%$ per month. Calculate the monthly payments.

2. Calculate the monthly payment in each case.

Sales	Down Payment	No. of Monthly Payments	Service Charge rate (monthly)
(a) $350	$80	6	$1\frac{1}{4}\%$
(b) $640	$40	12	$1\frac{1}{4}\%$
(c) $640	$40	6	$1\frac{1}{4}\%$
(d) $640	$40	12	$1\frac{3}{4}\%$

3. Compare the answers to questions 2(b) and 2(c). Also compare the amount of interest in each of these cases. Does it pay to make larger payments? Why?

4. Compare the payments in questions 2(b) and 2(d). Does a difference of $\frac{1}{2}\%$ per month make much difference in payments?

5. When making deferred payments what factors should you check to keep total payment as small as possible?

6. Why do you think companies like to advertise their interest (carrying or service) rates *per month* rather than *per annum*?

4·6 True Interest Rate in Instalment Buying

In the previous section we simply calculated the interest (apparent) on the basis of the total balance. This does not give us a true picture of the interest rate since the purchaser does not have all the money for the length of the contract. Study the example.

Example 1

Majorie and Phil purchase a TV for $560. There is a sales tax of 5%. They make a down payment of $108 and pay the balance plus carrying charges in 6 monthly payments of $82.40. Calculate the apparent and true interest rates.

Solution

Cost of TV	$560
Sales tax (560 × .05)	$ 28
Gross sales	$588
Down Payment	$108
Balance	$480

Total payments: $ 82.40 × 6 = $494.40

Interest charged:	$494.40
	$480.00
	$ 14.40

To calculate apparent interest rate:

$I = Prt$ (*interest rate per year*, *time in years*)

$14.40 = 480 \times r \times \frac{1}{2}$

$r = 6\%$

To calculate the TRUE INTEREST RATE is more difficult.

The two people actually borrowed $480. This they paid back in 6 payments. Hence amount of this paid back each payment is $\frac{480}{6}$ or $80.

They had $480 for 1 month. (Then they paid $80.)
They had $400 for 1 month. (Then they paid $80.)
They had $320 for 1 month. (Then they paid $80.)
They had $240 for 1 month. (Then they paid $80.)
They had $160 for 1 month. (Then they paid $80.)
They had $80 for 1 month. (They paid final $80.)

They had $1680 for 1 month.

They paid $14.40 interest.

To find interest rate:

$I = Prt$ (*rate per annum*, *time in years*)

$14.40 = 1680 \times r \times \frac{1}{12}$

$r = 10.3\%$ (rounded to the nearest tenth).

The TRUE RATE is 10.3% per annum.

This is a great deal different from the 6% per annum advertised!

EXERCISE 4·6

1. Harold purchased a new suit for $124. He paid $24 down and 5 monthly payments of $20.42. Calculate the interest he paid.
   Calculate the apparent and true interest rates.

2. Calculate the apparent and true interest rates in each:

Total cost	No. of monthly payments	Amount of each payment	Down payment
(a) $400	6	$64.08	$88
(b) $542	7	$69.63	$80
(c) $1200	12	$89.60	$290

3. Why do firms like to use the apparent rate when advertising their "low interest rates"? Discuss.

4. Most provinces require that all lenders of money provide their borrower with the cost of borrowing the money, both in dollars and cents and the annual interest rate. Why do you think governments feel this law is necessary? Discuss.

5. Information on Consumer Protection is available from both your Provincial and Federal Departments of Consumer Affairs. Information is also available from Information Canada, in the provincial capitals and in Ottawa. Write to each asking them for information.

   What does the Consumer Protection Act say about *cooling-off period* on sales made by salesmen calling at your home?

   What are the regulations regarding repossession of goods?

4·7 Formula for True Interest Rate

We shall now study a short cut for calculating an approximation of the true interest rate.
The formula is

$$r = \frac{2NI}{P(n+1)} \times 100$$

r = true interest rate
N = Number of interest periods per year
  If monthly payments, $N = 12$
  If quarterly payments, $N = 4$
  If yearly payments, $N = 1$, etc.
I = interest in dollars
P = principal; amount of loan that borrower receives
n = no. of equal payments

Example 1

Calculate the true interest rate on a loan of $300 if there are 6 payments of $55 quarterly.

Solution

Total of 6 payments:

$$\$55 \times 6 = \$330$$
$$\text{Loan} = \$300$$
$$\text{Interest} = \$\ 30$$

$$r = \frac{2NI}{P(n+1)} \times 100$$

$N = 4$ (since payments are made quarterly)
$I = \$30$ (interest in dollars)
$P = \$300$ (amount of loan)
$n = 6$ (number of payments)

$$r = \frac{2 \times 4 \times 30}{300(6+1)} \times 100$$
$$= \frac{240}{2100} \times 100$$
$$r = 11.4\% \text{ (slide rule accuracy)}$$

The true rate of interest is 11.4%.

The customer should be careful when borrowing money. Some companies advertise low payments and cost on loans but in fact do not make clear how their loan system works. One company used to advertise:

COMPARE OUR COST AND PAYMENTS

Borrow $500 for 12 months
Payments only a low $41.67 monthly
Interest costs only a low $28.

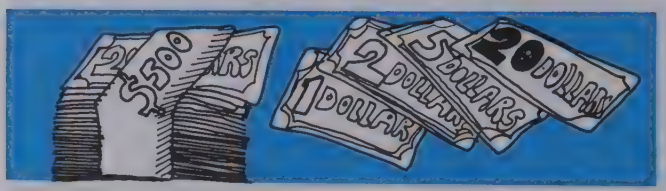
BUY THAT TV TODAY
SEE US FIRST

ABC FINANCE CO.

So a customer signs the contract (without reading it) and the loans manager says: "Here is your $500. But we will take the interest off this first so you don't need to worry about it. Therefore you get $472. It has been nice doing business with you."

Certainly the loans manager likes doing business with that kind of customer. The customer did not read the contract. He did not ask how much money he would actually get. He was a "sucker".

Don't you be a "sucker". Know all the details about your loan before you sign the contract.



Example 2

Calculate the apparent interest rate the company would expect the "sucker" customer to calculate for the loan advertised above. Calculate the true interest rate.

Solution

The ABC Finance Company hopes the "sucker" will calculate interest this way:

$500 for only $28

$100 for only $5.60

$$\begin{cases} I = prt \\ 28 = 500 \times r \times 1 \\ r = 5.6\% \end{cases}$$

WOWEE! Only 5.6% interest rate!!

But the true interest rate:

Since the interest is deducted from the $500, the borrower gets only $472. Hence $P = 472$.

The true interest rate:

$$r = \frac{2NI}{2P(n+1)} \times 100$$
$$= \frac{2 \times 12 \times 28}{472 \times 13} \times 100$$
$$= 10.9\% \text{ (slide rule)}$$

The true interest rate is 10.9% – NEARLY DOUBLE THE APPARENT RATE!!

Note!

Even the true apparent rate is more than 5.6%.
Actual amount of loan,

$$\$472 = P$$
$$I = Prt$$
$$28 = 472 \times r \times 1$$
$$r = 5.95\% \text{ – nearly } 6\%$$

EXERCISE 4·7

1. Calculate the true interest rate on a loan of $400 if there are 12 monthly payments of $45.

2. An ad like this appeared in a leading newspaper.

The Friendly Finance Company

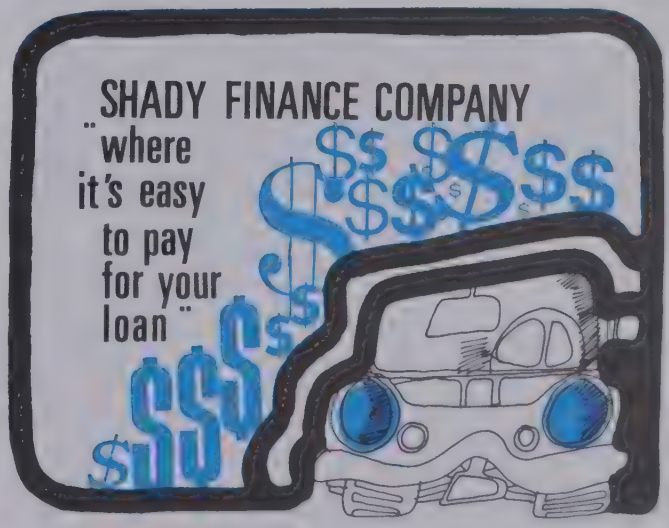
431-52 Street, Yorkville

INVITES YOU TO
COMPARE OUR LOW FINANCE TERMS

12 MONTHS			18 MONTHS			24 MONTHS			30 MONTHS		
LOAN	COST	PAYT	LOAN	COST	PAYT	LOAN	COST	PAYT	LOAN	COST	PAYT
50	3.16	4.43	50	4.72	3.04	50	6.40	2.35	50	7.90	1.93
100	6.44	8.87	100	9.62	6.09	100	12.80	4.70	100	16.10	3.87
150	9.72	13.31	150	14.52	9.14	150	19.44	7.06	150	24.30	5.81
200	13.00	17.75	200	19.42	12.19	200	25.84	9.41	200	32.50	7.75
250	16.16	22.18	250	24.32	15.24	250	32.48	11.77	250	40.40	9.68
300	19.44	26.62	300	29.22	18.29	300	38.88	14.12	300	48.60	11.62
350	22.72	31.06	350	34.12	21.34	350	45.28	16.47	350	56.80	13.56
400	26.00	35.50	400	38.84	24.38	400	51.92	18.83	400	65.00	15.50
450	29.16	39.93	450	43.74	27.43	450	58.32	21.18	450	72.90	17.43
500	32.44	44.37	500	48.64	30.48	500	64.96	23.54	500	81.10	19.37
550	35.72	48.81	550	53.54	33.53	550	71.36	25.89	550	89.30	21.31
600	39.00	53.25	600	58.44	36.58	600	78.00	28.25	600	97.50	23.25
650	42.16	57.68	650	63.34	39.63	650	84.40	30.60	650	105.40	25.18
700	45.55	62.12	700	68.24	42.68	700	90.80	32.95	700	113.60	27.12
750	48.72	66.56	750	72.96	45.72	750	97.44	35.31	750	121.80	29.06
800	52.00	71.00	800	77.86	48.77	800	103.84	37.66	800	130.00	31.00
850	55.16	75.43	850	82.76	51.82	850	110.48	40.02	850	137.90	32.98
900	58.44	79.87	900	87.66	54.87	900	116.88	42.37	900	146.10	34.87
950	61.72	84.31	950	92.56	57.92	950	123.28	44.72	950	154.30	36.81
1000	65.00	88.75	1000	97.46	60.97	1000	129.92	47.08	1000	162.50	38.75

(a) Calculate the true interest rate on those loans in rectangular boxes. Is the interest rate lower for short term loans or longer term loans?

(b) Calculate the true interest rate on those loans in oval frames. Is the interest lower for smaller loans or larger loans?



3. Mr. and Mrs. Brost were in urgent need for money to pay for car repairs. They visited a shady finance company and obtained a loan of $360. They signed a contract to pay 9 monthly payments of $40. The loan shark deducted the $30 interest charges and gave the Brosts $330. Calculate the true interest rate on the loan. Do you think this rate is exhorbitant? Do you think the Brosts were wise?

4·8 Sources of Money When You Need To Borrow

Generally, there are three places one might go to borrow small amounts of money. Each has advantages and each has some drawbacks.

A Banks

B Credit Unions

C Finance Companies

Project

Project 1

Make arrangements to visit, or to have a visit from a manager of a bank, of a credit union and of a finance company. Ask each to explain how their loan business operates.

Topics you may wish to have discussed are:

(a) Why would one do business with this firm over the others?

(b) What group of people are his customers (low wage earners, middle wage earners, etc.)

(c) Interest rates.

(d) Security and collateral required. (What do these words mean?)

(e) References, co-signers for minors, etc.

(f) Who can get a loan?

(g) How big a loan is available?

Prepare a list of questions for each manager.

EXERCISE 4·8

1. If you do not live in an area where you can carry out part, or all of the project, write a letter to the managers of those businesses you could not visit. Ask your questions in the letter.

4·9 Cost of Operating a Car

Many students operate their own cars or wish they had their own cars. But many do not really know what expenses are involved. Let us look at the cost of operating a car.

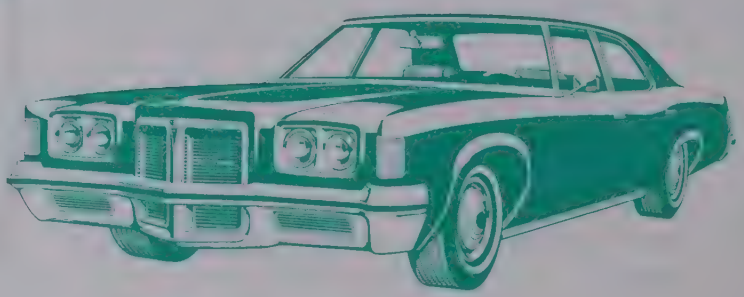
We will assume you drive about 10,000 miles a year. The car cost $2,500 (1-year old used car). The plans are to keep the car 2 years. You live in an urban centre.

	Yearly Cost
Depreciation (15% of cost of car)	$375
Insurance (under 25 years of age)	$200
License and registration	$20
Gas and Oil	$300
Maintenance	$150
Tires	$50
Parking, etc.	$30
Finance charges (11%)	$137
	$1262

In addition to this total cost of $1262, you must make the payments on the loan. The only cost that does not require an outlay of cash is the $375 for depreciation. Nevertheless, the car is costing about $100 a month operating expenses. Can you afford to operate a car?

It is recommended that a family spend no more than 15% of its income on transportation.

The depreciation on a car is the largest single expense. On a new car the depreciation is 25% of its value the first year; 15% the second and third year, then drops slightly per year after that. One way to save on high depreciation cost is to buy a demonstrator model or a good used car. Another way is to



keep the car for a longer time, spreading the depreciation cost over many years. As a car gets older its maintenance costs go up.

EXERCISE 4·9

1. If you own a car, keep a record of your expenses (including payments, if any) for 1 month. On this basis, what would be the yearly expense?

2. If it cost $900 to operate a car, and this is 28% of a young person's income, what is his income?

3. (a) If it cost $1200 a year to operate a car, and if this is 15% of the family income, what is the family income.

   (b) Do you think very many newly married couples can afford a car?

   (c) Many young married couples have cars. How do you think they can afford it since $1200 is likely more than 15% of a young man's income?

4. Using the figures given in this section, what is the average cost per month of operating a car?

5. Discuss advantages and disadvantages of buying:

   (a) a new car

   (b) a demonstrator

   (c) a second hand car.

6. Discuss the ways one can cut costs of operating a car.

7. Discuss the advantages of doing your own work on a car.

8. Discuss the advantages and disadvantages of paying cash for all car expenses rather than using a credit card or credit at a service station.

9. Phone 5 employers in your area to obtain the average starting salary for a young man. Find the average of the 5 starting salaries. What amount could be spent on a car by a person earning the average. (Use the guideline of 15%.)

Projects

Project 2

Invite a service station manager to speak to your class on the most common causes of high maintenance cost of cars. Also prepare any other questions you might want to ask him.

Project 3 (for boys mostly)

Visit the Automotive Servicing classes in your school or technical institution to learn the types of easy car servicing the average person can do.

Project 4 (for girls only)

Arrange to have a service station attendant speak to you on things every girl should know about the operating of a car.

4·10 Savings Accounts

Every person with an income, be it an allowance or a wage for full or part time work, should set up a budget. In this section we shall deal primarily with full-time independent wage earners. Statements made generally are applicable to students also.

A budget usually consists of four main categories:

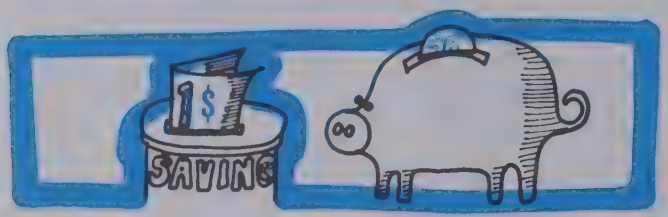
Income – usually fixed (if employed full time).

Fixed expenditures – cost of housing, insurance, licenses, etc.

Variable expenditures – recreation, food, clothing, automobile expenses, medical expenses, etc.

Savings – money set aside for emergencies, or for future planned events.

There are a number of manners in which one can institute a savings program. The most simple, but not to be recommended, method is the piggy bank or coffee jar.



Highly recommended savings methods can be arranged through banks, trust companies, credit unions and government investment certificates.

Savings accounts should be established on the basis whereby fixed amounts are deposited regularly. Certain advantages are available from each institution. Banks, credit unions and trust companies pay interest on the money deposited in savings accounts. The rate varies from time to time and is fixed by the individual institution. In some cases interest is paid on the minimum monthly balance but is credited to the account quarterly, semi-annually or annually. Chartered banks in Canada calculate interest on the minimum quarterly balance and add it to the account on October 30 and April 30.

A typical pass book on a savings account is shown.

NATIONAL BANK BRANCH 25099-0011
SAVINGS ACCOUNT
ACCOUNT NO.

DATE	ITEM	WITHDRAWAL		DEPOSIT		BALANCE	
Apr. 30	FWD					562	70
30	INT			4	52	567	22
May 15		225	22			342	00
July 30				332		674	00
Aug. 16				22	50	696	50
Oct. 21				200	50	897	00
Oct. 30	INT			12	69	909	69

Notice the interest has been recorded on April 30 and October 30. The interest on October 30 is accumulated from two 3-month periods.

The minimum quarterly balance from May 1 to July 30 was $342.

Interest for this period is

$$342 \times .05 \times \tfrac{1}{4} = \$4.27$$

The minimum quarterly balance from August 1 to October 30 was $674.

Interest for this period was

$$674 \times .05 \times \tfrac{1}{4} = \$8.42$$

Total interest credited to the account on October 30 is $12.69.

One should also note that the interest is compounded. That is, interest is calculated on the interest added earlier.

Credit Unions

Credit unions are organized by the people for the people as a non-profit organization. Usually credit union organizations are owned and operated by their members.

The basic purposes of a credit union are to provide its members with higher interest rates on deposits and lower interest rates on loans than is normal in other comparable situations.

In order to take advantage of credit union services one must be a member of the organization. The members often have a common interest such as occupation. To be a member one must buy shares in the organization.

Government Bonds

The Canadian Government issues bonds for sale each November. Only Canadians and Canadian organizations can purchase these bonds. The bonds pay a high interest rate if left to maturity. Canada Savings Bonds come in $50, $100,$500, $1000 and $5000 denominations. The bonds can be cashed at any time but for less than full maturity value. Since 1967 a customer can buy $100 and 13 years later receive $200 in return.

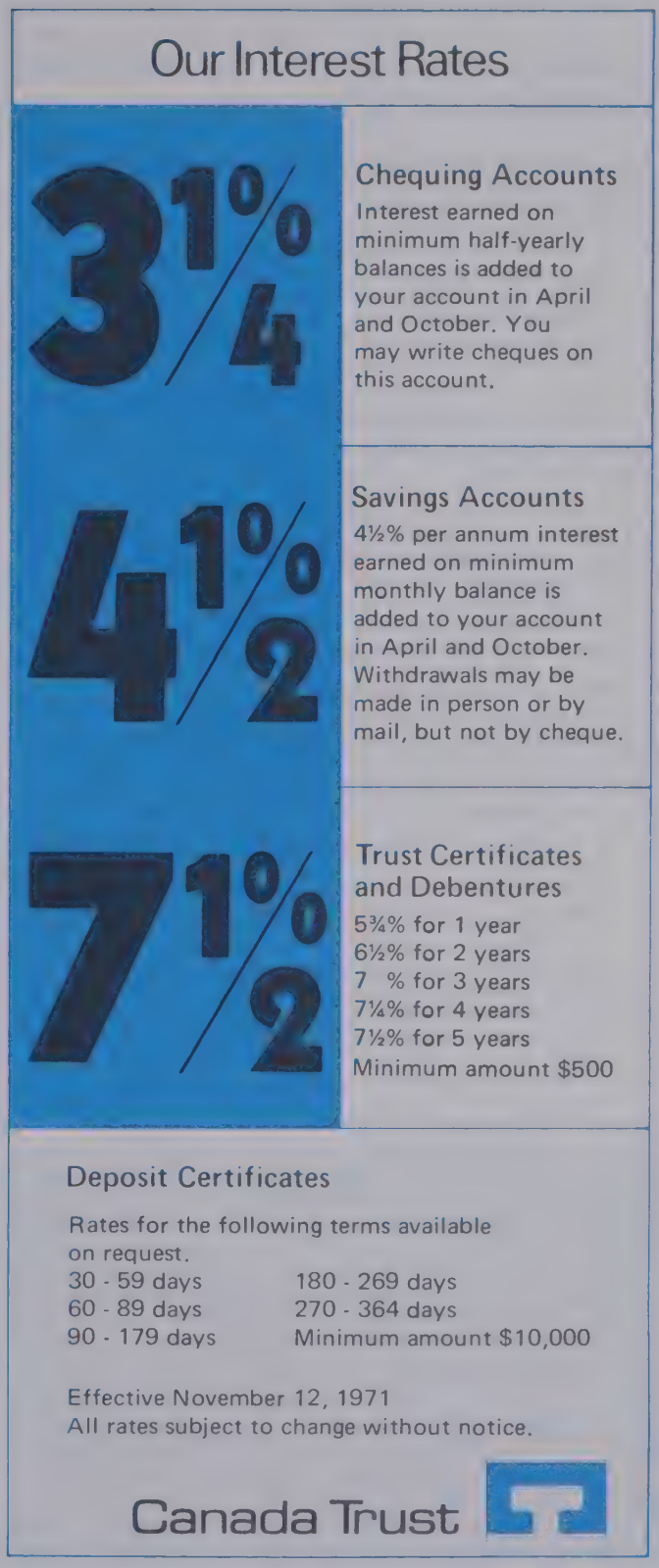
A customer may buy bonds from a bank or stockbroker, or under a Payroll Savings Plan (deductions are made regularly from one's pay).

EXERCISE 4·10

1. The ad on this page appeared in a local newspaper. Calculate the interest on $500 for one year at the different rates. Explain under what circumstances a person would invest his money in each plan. Discuss each with your class.

2. Obtain from a bank or trust company suggestions for setting up a budget. Use the information to set up a budget for a family of four (father, mother and 2 children under 6). Assume the family income is
   (a) $8000 gross
   (b) $10,000 gross.

3. What are the disadvantages of saving in the piggy bank or coffee jar?


Our Interest Rates
3¼%
Chequing Accounts
Interest earned on minimum half-yearly balances is added to your account in April and October. You may write cheques on this account.
4½%
Savings Accounts
4½% per annum interest earned on minimum monthly balance is added to your account in April and October. Withdrawals may be made in person or by mail, but not by cheque.
7½%
Trust Certificates and Debentures
5¾% for 1 year
6½% for 2 years
7 % for 3 years
7¼% for 4 years
7½% for 5 years
Minimum amount $500
Deposit Certificates
Rates for the following terms available on request.
30 - 59 days
60 - 89 days
90 - 179 days
180 - 269 days
270 - 364 days
Minimum amount $10,000
Effective November 12, 1971
All rates subject to change without notice.
Canada Trust

4. What are the advantages of saving through a bank plan? A credit union plan? A trust company plan? Government certificates or Canada Savings Bonds?

5. Complete the entries in the pass book. Interest rate is 5%.

NATIONAL BANK BRANCH 25099-0011
SAVINGS ACCOUNT
ACCOUNT NO.

DATE	ITEM	WITHDRAWAL		DEPOSIT		BALANCE	
Nov. 30	FWD					350	-
30	INT			2	92	352	92

Mr. Brown withdrew $35 on December 15 and $42 on December 20; on January 30 he deposited $32.50. On February 15 he withdrew $12. On March 30 he deposited $34. On April 20 he deposited $43.25. On May 25 he deposited $12.

Bring Mr. Brown's pass book up to date.
Hint! Don't forget to enter his interest on April 30.

6. Make up a pass book for this information:

May 1	forward $72.50		
Deposits:		Withdrawals:	
May 16	$13.25	May 20	$18.70
30	5.60	July 2	12.50
June 13	42.50	Sept. 12	32.50
July 12	6.15	Nov. 23	6.65
Aug. 15	33.70		
29	63.00		
Oct. 15	14.90		

Projects

Project 5

Arrange to visit a bank to learn about savings accounts, or arrange to have a bank manager or his assistant speak to you on the aspects of savings accounts.

Ask the manager to show you how to complete deposit and withdrawal slips. Also ask him to give you information that you, as a customer, should know about savings accounts in order for you to make the most of a savings account.

Prepare a list of questions before the visit of the bank manager. You may wish to have the bank manager sit as a member of a panel. Select 3 from your class to act as other panel members. Preparation is necessary on the part of panel members in order to make a positive contribution. Panel members will ask the questions which the bank manager will answer.

Ask the manager to provide information on Canada Savings Bonds. Ask about the advantages and disadvantages over other forms of savings of (a) banks savings accounts, (b) Canada Savings Bonds.

Since some members of your class may be interested in banking as a career ask the visitor to tell you about the nature of the work in a bank.

Project 6

Arrange for the visit of a Credit Union manager or accountant to answer questions similar to those posed to the bank manager.

Ask how you can become members of a credit union.

Ask about advantages and disadvantages of a credit union.

Project 7

Write the Royal Bank of Canada for their excellent material entitled *Your Money Matters*. This is a set of 6 booklets and overhead transparencies. Also ask for the free loan of their films that accompany the above materials:
(a) *Your Money Matters*
(b) *More Than a Matter of Money*.

Project 8

Make up a Bulletin Board display related to savings. You may wish to divide the board into sections such as:

Bank	Credit Union	Government Bonds	Others	Reasons for Saving

Note to teacher: To meet needs of individual students, divide the class into groups, each group to visit one of the places indicated. Each group will prepare a display and presentation. Provide a question and answer period.

4·11 Insurance

The purpose of insurance is to prepare for help in advance of money losses arising from risks and dangers of day to day living. Car insurance is purchased to pay for losses one might incur from accidents, fire, theft, vandalism or damage to other people or their property.

Insurance on personal property—one's house, furniture, etc. is purchased to pay for any loss (usually only partial loss) suffered from any damage or loss that might occur, i.e. fire, flood, storms, theft, etc.

Life insurance is purchased to pay a beneficiary a set amount of money in lump sum or in periodic payments upon the death of the insured. Coupled with life insurance policies are disability policies which pay certain amounts if the insured is unable to work (insured against loss of income due to ill health).

How does an insurance company operate? Insurance companies operate on *"average" expectancy.* Let us consider life insurance. From records that have been kept over long periods of time, a company knows how many people on the average will die between ages 20 and 29, and how many will live to various ages, 40, 50, 60, etc. Based on this information the company calculates the average amount that will have to be paid out in *claims* and the amount that will have to be collected in *premiums.* The premiums on any one policy can then be calculated.

An insurance company must have, at any one time, an amount of money with which to pay off claims if and when it is necessary. It is totally unlikely that all insurance policies will become payable at any one given time. The company has the extra money *invested* in many types of investments which provide the company *revenue* from interest on the investments. Because the company has the use of the insured's money for many years and the company collects interest on this money, the insured does not necessarily pay in as much as is paid out on his behalf.

Let us look at three types of life insurance policies.

Whole Life Insurance

At the age of 25 John Petta takes out a life insurance policy for $10,000 payable upon his death. He must pay the premium each year that he lives, or he may elect to pay a higher premium for a fixed period (15, 20, 30 years) and receive lifetime protection. On his death, be it in his 26th year or his 105th year, his beneficiary receives $10,000.

Endowment Policies

These policies, while providing regular life insurance during the term of the policy, also build up a definite savings during a given number of years. At the end of the period, if the insured is still living, the life insurance policy is terminated and the insured receives a specific payment (cash value). These policies usually cost more annually than other types of life insurance.

Term Insurance

This type of policy provides protection for a certain period of time, usually 5, 10, or 20 years. At the end of the *term,* all protection ceases with no cash payment. Usually the insured can renew but at a higher premium as the policy holder is older.

Insurance Booklet

The teacher may write for classroom copies of each booklet. This should be done well in advance of the time the class will use this unit.

"My husband has everything, and I thought a policy to cover it all would be nice."

Project

Project 9

Write Educational Division, The Canadian Life Insurance Association, 44 King Street West, Toronto 1, Ontario.

Ask for these booklets:

1. The Story of Life Insurance
2. A Miss and Her Money (for girls only)
3. Money in Your Pocket (for boys only)
4. The Family Money Manager
5. You and Your Family's Life Insurance
6. Careers in Life Insurance
7. Actuary
8. Canadian Life Insurance Facts
9. Life Insurance: Dollars and Sense
10. A Career for You in a Life Insurance Company.

Ask for 1 copy of each of these film strips for your school:

1. Careers in Canadian Life Insurance Underwriting
2. The Life Insurance Story Parts I, II and III.
3. Life Insurance – a Canadian Handbook

Use the above materials to answer these questions:

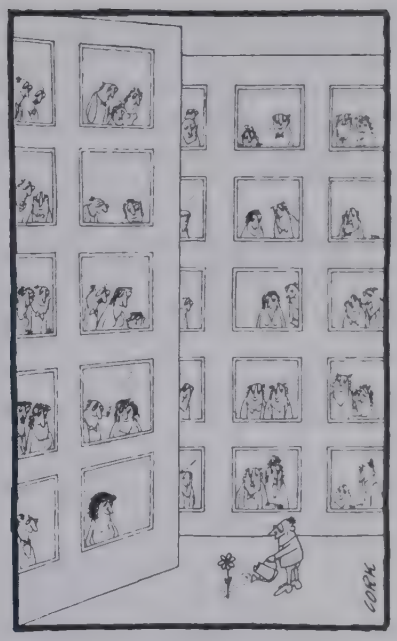
1. What is a mortality table?
2. What is the origin of insurance?
3. What is the origin of life insurance in Canada?
4. How are benefits paid?
5. What is an annuity?

In the last 20 years, how many homes in Canada have been financed by life insurance funds?

The answer is on the next page.

6. How do registered retirement savings plans affect one's income tax?
7. What does a life insurance company do with all the money it collects in premiums?
8. What is the difference between a *non-participating* and a *participating* policy?
9. Discuss how a *family income plan policy* operates.
10. What is a *disability waiver of premiums* benefit?
11. What kind of policies can you use to borrow money?
12. Use the budget sheet in *The Family Money Manager.* Prepare the budget for a man and his wife, if only the man works and his total annual income is $4760. Estimate what you think are realistic figures. Do not operate in the red.
13. Why do you think the Life Insurance Companies ran this ad in a national magazine?

What does this cartoon say about high rise living?



Since 1950 the mortgage and real estate investments of life insurance companies have financed more than a million homes for Canadians. The life insurance companies place into housing about half of all money invested for policyholders.

The Life Insurance Companies in Canada.

EXERCISE 4·11

Let us suppose that the annual cost of $1000 of a whole life policy is:

$12 at age 15	$13 at age 20
$15 at age 30	$17 at age 40

30-pay-life policy is:

$17 at age 15	$19 at age 20
$22 at age 30	$25 at age 40

1. When Herb was 15 his father purchased a $5000 whole life policy on Herb.
   (a) What is the yearly premium?
   (b) If Herb dies at the age of 40, how much would have been paid in premiums?
   (c) How much would the beneficiary receive?

2. At the age of 20, Phil took out a 30-pay-life policy for $10,000 (he only makes 30 annual payments but is insured for life).
   (a) What is his annual payment?
   (b) How much is paid to Phil's beneficiary if Phil dies at age
      (i) 21?
      (ii) 40?
      (iii) 60?
   (c) Why are premiums for a 30-pay-life policy more than for a whole life policy?
   (d) Why are premiums for a man of 40 more than for a man of 20?

Let us suppose that the annual cost of $1000 of term insurance is:

$5.15 at age 18
$6.25 at age 25
$7.90 at age 30
$8.30 at age 35
} term to age 60

3. At age 25, Mark took out a $25,000 term to age 60 policy, naming his wife beneficiary.
   (a) What was his annual premium?
   (b) How much does his wife receive if he dies at age
      (i) 26?
      (ii) 35?
      (iii) 61?
   (c) Mark died at age 55. His widow invested the death benefits in $6\frac{1}{4}\%$ bonds. What monthly income will she receive?
   (d) Do you think the widow can live on the income from the bonds?
   (e) How much had Mark paid in premiums when he died?
   (f) If Mark's income had been $6000 a year, what amount would have to be invested at $6\frac{1}{4}\%$ to produce an income of half his salary?

4. At the age of 35, Mark had taken out an additional $10,000 term to 60.
   (a) What was the annual premium for this policy?
   (b) Why does a young insurance buyer pay less for a policy than an older buyer?
   (c) Mark's widow invested the death benefits from this policy at 5%. What was the monthly income from this investment?

The premium on an insurance policy is based on the age of the insured at the time the policy is taken out. This chart indicates the annual premium for $1000 of a 15-year endowment policy at various ages:

$56.25 at age 1
$55.25 at age 15
$57.75 at age 20
$56.10 at age 30

5. What is the annual premium for a person taking out a $2000 15-year endowment policy at age:
   (a) 1 year
   (b) 15 years
   (c) 30 years

6. Harris and Jill bought a 15-year endowment policy for $5000 on their son Harris Jr. at the age of one. They planned that the $5000 could be used to meet the cost of his college education.
   (a) What premium is paid on the policy?

(b) What was the total Harris and Jill paid in premiums if Harris Jr. received a cash payment on his 16th birthday?

7. (a) Harris also took out a 15-year endowment policy for $5000 when he was age 30. What was the premium?

(b) Why do you think the premium is higher for a child aged 1 than for a person aged 15 or 30?

(c) Harris lived to benefit from the endowment. What benefit did he collect on maturity of the plan?

(d) Harris paid both premiums in monthly payments. To do this he must pay an additional 1% on each payment. What was his monthly premium?

4·12 Property Insurance

Each home owner should insure both his home and the contents including personal items such as clothes, books, etc.

Insurance rates on houses vary with the type of structure (solid brick, frame, cement block, etc.), the location, and availability and quality of fire-fighting equipment. Rates for fire insurance are usually quoted as a sum per $100 unit of insurance for a period of 3 years.

For a small extra premium extended coverage may be obtained. This covers damages due to windstorms, lightning, explosions, vandalism, plumbing and heating malfunctioning, etc.

One does not include the value of the land when insuring a home. Why?

The contents of a house are usually insured separately.

"Know something? We're in debt beyond our wildest dreams!"

3 Year Premium Rates per $100 of Insured Value For a Single Family Dwelling

Location	Type of Construction: Brick			Brick Veneer			Frame		
	Fire	Fire and Extended Coverage	Contents	Fire	Fire and Extended Coverage	Contents	Fire	Fire and Extended Coverage	Contents
Farm (more than 5 miles from Fire Hall)	.72	.79	.68	.76	.83	.73	.81	.89	.78
Small town (minimum fire equipment)	.43	.49	.42	.47	.54	.44	.52	.60	.49
City	.33	.38	.30	.38	.44	.35	.43	.52	.39

When two companies insure the same property each does not pay the full loss. They pay a portion of the loss in proportion to the protection each provides.

Example 1

Calculate the insurance premium for fire and extended coverage on a brick veneer house valued at $23,000 in a small town. Contents valued at $5500 are also insured.

Solution

Extended coverage rate on brick veneer house in a small town $ .54/$100/3 years

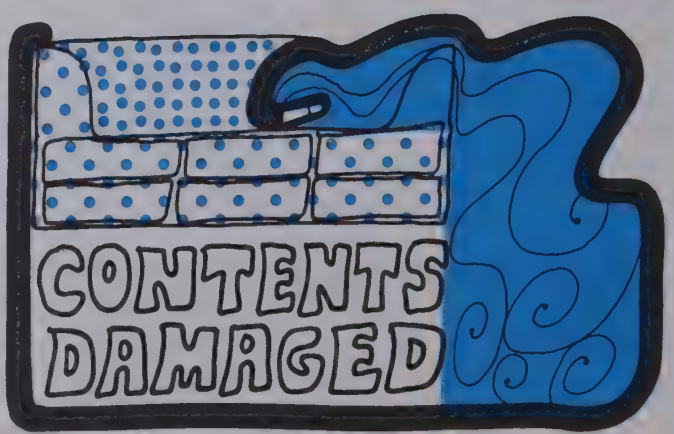
Value of house = $23,000
Rate = .54

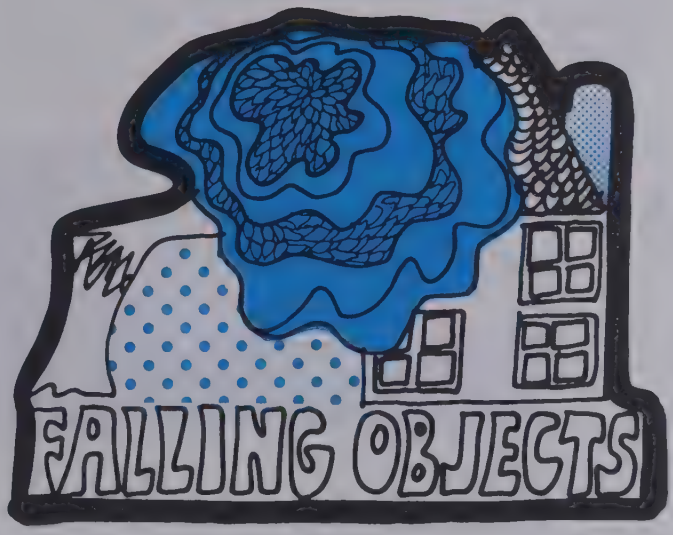
Premium = (value in $100's) × rate
= $230 × .54
= $124.20

Premium on contents
= (value in $100's) × rate
= $55 × .44
= $24.20

Total premium = $124.20 + $24.20
= $148.40

A home owner often takes out additional coverage to protect him against legal liability if one is injured on his property. For instance, if a person slips and falls and injures himself on the icy steps of a home (and if proper precautions were not taken to keep the steps clear of ice) the injured person may sue the home owner.





EXERCISE 4·12

Use the rates given in this section.

1. Calculate the premium for fire and extended coverage on a frame house on a farm valued at $18,500 and on the contents valued at $4350.

2. Mr. Morris purchased a fire insurance policy. His brick veneer house in a suburb of a large city was valued at $14,500. The contents were valued at $3,890. Calculate the premium for fire and content coverage.

3. Calculate the total premium on Mr. Morris's house and contents for fire and extended coverage and content coverage.

4. How much does the extended coverage cost Mr. Morris over the straight fire coverage?

5. Mr. Williams contemplated buying a fire policy on his $24,700 brick house in a residential section of a city. What is the difference in premiums between a straight fire policy and a fire and extended coverage policy? Is there an easy way to calculate this difference?

6. Considering the cost do you think Mr. Williams should take the extended coverage? Why?

7. Very often a person does not insure his home for total value.

   Mr. J. V. Norris insured his $36,500 brick home against fire for 90% of its value, and on 85% of the contents valued at $12,700. Mr. Norris lives in a small town. He also carried legal liability coverage for $100,000 at $ .30 per $1000 coverage. What is his total premium if he also takes extended coverage?

8. Why would one not insure his house for full value in a city?

9. A cottage at a lake is insured for $12,600. The contents are valued at $2200. What is the premium for fire and extended coverage and content coverage?

Project

Project 10

Consult a local insurance agent in your community and ask for the cost of a typical homeowner's policy that includes fire and extended coverage, coverage on contents, legal liability and extra living expenses if the homeowner cannot live in house because of damage.

Prepare a list of questions before your visit with the agent.

4·13 Automobile Insurance

Why should one carry insurance on his car? There are two basic reasons: (1) to pay for the damage to himself and his car in the event of an accident; (2) to pay for the damage to other people's property or to meet the medical bills or lawsuits arising from injury to other persons.

Four main types of automobile insurance will be discussed briefly here.

Third Party Liability

Most provinces require that car owners carry this insurance or provide proof of financial responsibility. The amount of coverage required varies from province to province. Third Party Liability is made up of two parts:

Public —This protects the owner against claims arising from his liability for injury or death resulting from an accident involving his car, up to the face value of the policy.

Property —This covers the owner's liability for damage to the property of others. This pays for the owner's liability in the damage to the owner's car in an accident. The owner may reduce this premium by assuming some of the risk, e.g. the first $100 of damage. This is known as a *deductible* clause. The higher the deductible, the lower the premium.

Comprehensive Insurance

This covers damage to the owner's automobile caused by fire, theft, windstorms, glass breakage and vandalism. It does not cover damage resulting from collision.

Crash tests show autos too flimsy

Crash tests of several 1970 car models were conducted by the Insurance Institute for Highway Safety in the United States. Following is a table of the damage resulting, with repair costs estimated by Canadian appraisers:

	5 m.p.h. front end into barrier	15 m.p.h. front end into barrier	10 m.p.h. front-to-rear crashes identical cars total cost both cars	10 m.p.h. front-to-side identical cars total cost both cars
Impala	$262	$ 975	$ 547	$659
Galaxie	233	946	634	587
Fury	207	794	593	670
Ambassador	430	1,141	1,125	695
Camaro	167	1,344	595	573
Mustang	221	869	813	535
Barracuda	276	1,126	479	532
Javelin	341	956	795	701
Volkswagen	128	635	247	467
Toyota	193	757	415	553
Maverick	206	793	574	525
Hornet	277	882	775	705

Discuss this newspaper item. What does it tell you about cars? Would this table help you to decide what new car to buy? What valuable information for the driver is not given?

Often fire and theft are sold separately.
Usually a $25 deductible clause is included to keep the premium down.

Medical Payments

This insurance covers the cost of medical expenses or funeral expenses of any occupant of the insured automobile.

A car owner should read his policy very carefully and get a full explanation of all the coverages, and of all those things not covered by his policy. This may save him a great deal of misery and regret at a later time.

Automobile Insurance Premium

The premium charged will depend on a number of factors:

1. Your location – rates in various parts of the country differ. Usually urban areas with high population density have a higher rate than rural areas. Some cities have higher rates than other cities.
2. Classification of driver or drivers:
   (a) over the age of 25
   (b) principal driver over 25 – but car is also driven by a driver under 25.
   (c) the principal operator is under 25.

   Some insurance companies give females under the age of 25 the same rates as drivers over 25. Also, some companies give lower rates to students with honour standings in school.
3. The use of the car – a car used for pleasure only has a lower rate than a car used for business.
4. The type of automobile – large, new, expensive cars have higher rates than small or older cars.
5. The accident record of the driver.

The following chart gives some indications of the combinations:

Sample Premiums for 3 or more accident free years

Kind		Age of car: 1 year or less				Age of car: 4 years or over			
		Driver(s) over 25		Driver(s) over 25 and under 25		Driver(s) over 25		Driver(s) over 25 and under 25	
		Pleasure	Business	Pleasure	Business	Pleasure	Business	Pleasure	Business
Third Party	Limits $50,000	$41.40	$78.40	$208.00	$222.40	Rates same as previous categories – age of car not a factor			
	500,000	48.00	89.20	242.50	261.75				
Collision or Upset	Deductible $100	49.50	78.00	192.25	222.50	39.90	62.50	123.50	151.60
	200	32.25	59.50	135.15	191.00	27.75	49.60	86.00	124.00
Comprehensive	Deductible None	62.50	65.50	83.00	86.00	42.40	45.40	56.80	59.80
	$25	31.25	34.25	41.50	44.50	26.25	28.25	29.30	32.30
Medical	Limits $500	3.00	4.00	4.00	5.00	3.00	4.00	4.00	5.00
	2000	5.00	6.00	6.00	7.00	5.00	6.00	6.00	7.00

Example 1

Calculate the annual premium on a new car priced about $5000 for Third Party Liability, limit $500,000, Collision or Upset, $100 deductible, Comprehensive, $25 deductible, and Medical, limit $500, for a driver over 25 using the car for pleasure.

Solution

We will look in the shaded sections in order:

Third Party Liability	$48.00
Collision	$49.50
Comprehensive	$31.25
Medical	$ 3.00
Total	$131.75

The annual premium is $131.75.

EXERCISE 4·13

1. Do you think it is fair that drivers under the age of 25 have higher rates than drivers over the age of 25? Explain why you feel the way you do. Why do you think some companies treat all female drivers as though they were over age 25 but they do not treat males this way. Do companies in your area have a sex differential? Does a company in your area offer a reduced premium plan for students with high school standing?

   Invite an automobile insurance agent to discuss with your class automobile insurance. Prepare a list of questions for him.

   Use the rates shown in this section to do the questions 2 through 5. Find the annual premium on each car:

Age of car	2. new	3. over 4 years	4. new	5. over 4 years
Purpose	business	pleasure	pleasure	business
Age of Driver	over 25 and under 25	over 25	over 25	over and under 25
Third Party Liability	$500,000	$500,000	$500,000	$500,000
Collision or Upset	$100 deductible	$200 deductible	$100 deductible	not carried
Comprehensive	$25 deductible	not carried	no deductible	$25 deductible
Medical	$2000	$500	$2000	not carried

Using the Library

1. Write a report on the history of insurance. Endeavour to show the development of insurance as a response for a need for protection.
2. Obtain a *mortality rate* table. What information does this table provide?
3. Research and write a paragraph on the development of *credit buying*. Endeavour to show the need for credit buying.

REVIEW EXERCISE 4

1. Hard Luck Harry borrows $300 from the Easy-Money Finance Company. The manager says "You only need to pay $50 every three months for 18 months. Here is your $300 – now just to make things easy and simple I'll take the $50 interest. So here is your balance of $250."
   (a) What is the apparent simple interest rate?
   (b) What is the true rate of interest?

2. Under what circumstances might a person want to take out
   (a) a term insurance policy?
   (b) an endowment policy?

3. A house worth $20,000 is insured (unwisely) with two insurance companies for $16,000 each. Why do you think there are clauses in the policies that state both companies will not pay the full $16,000 each in the event of the loss of the house?

Chapter Five
RATIO AND PROPORTION

5·1 Ratio

Carbon monoxide is a pollutant produced by the car using regular gasoline. The same car operating on natural gas produces very little carbon monoxide.

Table 1 Grams of CO pollutant produced

	1 mi.	10 mi.	50 mi.	500 mi.	1000 mi.
reg. gasoline	16.20				
natural gas	0.54				

We are able to compare the number of grams produced by each type of fuel in propelling the car 1 mile.

A comparison of two numbers expressed as a fraction is called a ratio.

Very often the ratio can be expressed in lower terms:

$$\frac{16.20}{0.54} = \frac{30}{1}.$$

Another way to express a ratio is by means of a colon. For example,

$$16.20 : 0.54$$

is read

EXERCISE 5·1

1. Copy and complete Table 1.
2. Express as a ratio the comparison of the amount of pollutants formed in going the distances indicated.
3. Express each ratio in question 2 in lowest terms.
4. Express each ratio in question 2 in the form $a : b$.

The ratio of garbage produced by a city varies directly with the population. What should a large city do with its garbage?

Height of a tall building can be estimated by comparing the length of its shadow with the length of the shadow of a post of known height.

5·2 Ratio of Three Items

Natural gas costs 19.6 cents per 98 cubic feet. A standard car uses 98 cubic feet to go 14 miles. We can represent this by the 3-term ratio in this way:

19.6¢ per 98 cubic feet per 14 miles

19.6¢ : 98 cubic feet : 14 miles

or without units

$$19.6 : 98 : 14$$

A three-term ratio can be written
$a : b : c$
where not all three terms are zero.

EXERCISE 5·2

1. Regular gasoline costs 48¢ per gallon. A standard car uses 1 gallon to go 22 miles. Express this information as a ratio of 3 terms.

2. A certain mixture of concrete contains 10 cubic yards of cement, 20 cubic yards of sand and 40 cubic yards of gravel. State the ratio of cement to sand to gravel.

3. An inheritance was left to four beneficiaries named in a will. The widow received \$6,500, the daughter received \$10,000, the son received \$8,000 and the brother received nothing. Write the ratio to indicate the amount each of the four received.

5·3 Equivalent Ratios

Are the fractions $\frac{3}{1}$ and $\frac{45}{15}$ equivalent? How do you know? Recall from Section 2·8 the *Multiplication Property of 1*.

An example of this property is:

$$\frac{2}{3} = \frac{2}{3} \times \frac{4}{4}$$
$$\frac{2}{3} = \frac{8}{12}.$$

We say $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent fractions.

Equivalent ratios are very similar since ratios are fractions. Consider the ratio of the grams of CO produced by a car on regular gasoline to that of a car on natural gas in Section 5·1.

$\frac{16.20}{0.54}$ reduces to $\frac{30}{1}$.

We say the two ratios are equivalent. This equivalence is written with an equal sign.

$$\frac{16.20}{0.54} = \frac{30}{1}$$

Note that $\frac{30}{1} \times \frac{0.54}{0.54} = \frac{16.20}{0.54}$.

> Two ratios $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if for some number k $(k \neq 0)$
> $$ak = c \quad \text{and} \quad bk = d.$$

This definition can be extended to three or more term ratios in that

If $a : b : c = d : e : f$
then $ak = d, \quad bk = e$
and $ck = f.$

Example 1

Determine whether or not each pair of ratios is equivalent.

(a) $\frac{2}{7}, \frac{22}{77}$
(b) $3 : 8, \quad 13 : 32$
(c) $2 : 5 : 7, \quad 9 : 22 : 31\frac{1}{2}$

Solution

(a) Find k such that $2k = 22$. $k = 11$.
Now, does $7k = 77$?
$7 \times 11 = 77$
Yes, it does. Therefore the two ratios are equivalent.
$$\frac{2}{7} \times \frac{11}{11} = \frac{22}{77}$$

(b) Find k such that $3k = 13$. $k = 4\frac{1}{3}$.
Now, does $8k = 22$?
$8 \times 4\frac{1}{3} = 34\frac{2}{3}$.
NO! It does not, therefore the two ratios are not equivalent.

(c) Find k such that $2k = 9$. $k = 4\frac{1}{2}$.
Now, does $5k = 22$ and does $7k = 31\frac{1}{2}$?
Testing, does $5k = 22$?
$5 \times 4\frac{1}{2} = 22\frac{1}{2}$.
NO! Immediately, without even trying the other part, we know the ratios are not equivalent.

EXERCISE 5·3

Determine whether or not each pair of ratios is equivalent. Find k for each pair of equivalent ratios.

1. $\frac{2}{3}, \frac{8}{12}$
2. $\frac{3}{5}, \frac{15}{25}$
3. $\frac{5}{6}, \frac{12.5}{15}$
4. $\frac{7}{8}, \frac{91}{102}$
5. $\frac{15}{12}, \frac{4m}{3m}$
6. $\frac{42}{14}, \frac{3}{1}$
7. $\frac{55}{44}, \frac{11}{4}$
8. $3:11,\ 39:43$
9. $4:15,\ 16:45$
10. $36:24,\ 6:4$
11. $56:24,\ 8:3$
12. $96:16,\ 6:1$
13. $3:9:11,\ 12:36:44$
14. $6:7:8,\ 18:21:32$
15. $12:15:18,\ 36:45:98$
16. $8:10:11,\ 100:125:132$
17. $54:36:12:6,\ 9:6:20:0$
18. A cement mixture was to have had the sand to gravel to cement in the ratio of 5:4:2. The contractor put in 175 lb of sand, 140 lb of gravel and 70 lb of cement. Was this correct?

19. Mary, Jane, and Sue decided to share the money they earned in the ratio of 8:6:5. Mary received \$24, Jane \$18 and Sue \$20. Was this according to the agreement? Explain.
20. Mike, Tiny and Stretch respectively, scored 27, 18 and 9 points in the basketball game. Mike said their scores were in the ratio 3:2:1. Was this correct?

5·4 Proportion

> An equation formed by exactly two ratios is a proportion.

These are examples of proportions.

$$\frac{2}{3} = \frac{4}{6} \quad (1)$$

$$\frac{3}{5} = \frac{x}{35} \quad (2)$$

$$7:8 = 49:56 \quad (3)$$

$$6:7 = x:42 \quad (4)$$

Statement (3) is read: "7 is to 8 as 49 is to 56."

Statements (2) and (4) are true only under certain conditions. In (2) x must be 21. In (4) x must be 36. Do you see why?

Do you recall that we said two fractions are equal if the cross products are equal?

$$\frac{2}{3} \times \frac{4}{6} \qquad 12 = 12$$

Apply this same principle to proportion (2).

$$\frac{3}{5} \times \frac{x}{35} \qquad 5x = 105, \quad x = 21$$

> In a proportion the product of extremes equals the product of means.

extreme → 2 : 3 = 6 ← extreme
mean → 3 : 6 ← mean

extremes: 7 : 8 = 49 : 56
means: 7 : 8 = 49 : 56

Applying this principle to proportion (4) we have

$$6 : 7 = x : 42$$
$$7x = 252$$
$$x = 36.$$

Just as we agreed earlier!

Example 1

Is this a true proportion $\frac{4}{7} = \frac{56}{102}$?

Solution

Using the *extremes and means* method we have

$$\frac{4}{7} \times \frac{56}{102} \quad 392 \ ? \ 408$$

Since the two products are not equal we then know $\frac{4}{7}$ and $\frac{56}{102}$ are not equivalent therefore $\frac{4}{7} = \frac{56}{102}$ is not a true proportion.

EXERCISE 5·4

Determine whether or not each is a true proportion.

1. $\frac{3}{5} = \frac{15}{25}$
2. $\frac{4}{7} = \frac{14}{24}$
3. $\frac{13}{18} = \frac{142}{198}$
4. $\frac{4}{7} = \frac{30}{56}$
5. $\frac{11}{21} = \frac{21}{42}$
6. $\frac{12}{7} = \frac{96}{84}$

Find the value of the unknown that makes each true proportion.
Use the *extremes and means* method.

7. $\frac{3}{7} = \frac{n}{42}$
8. $\frac{4}{9} = \frac{n}{45}$
9. $\frac{8}{11} = \frac{n}{121}$
10. $\frac{n}{6} = \frac{40}{120}$
11. $\frac{n}{8} = \frac{81}{72}$
12. $\frac{n}{9} = \frac{144}{108}$
13. $\frac{3}{n} = \frac{24}{32}$
14. $\frac{4}{n} = \frac{65.1}{27.9}$
15. $\frac{14}{13} = \frac{32.2}{n}$
16. $12 : 7 = n : 56$
17. $16 : n = 40 : 50$
18. $n : 32 = 52 : 104$

5·5 Using Equivalent Ratios With Three Items

Example 1

Three basketball players made 96 points. The points made were in the proportion 3:6:7. How many points did each make?

Solution

Consider the points each made to be A, B, C.

We then have $3 : 6 : 7 = A : B : C.$

Therefore
$$\left.\begin{aligned} 3k &= A \\ 6k &= B \\ 7k &= C. \end{aligned}\right\} \ (1)$$

And since $A + B + C = 96$

then $3k + 6k + 7k = 96$

$$16k = 96$$
$$k = 6.$$

Making replacements in (1) we get

$3k = A$	$6k = B$	$7k = C$
$3 \cdot 6 = A$	$6 \cdot 6 = B$	$6 \cdot 6 = C$
$A = 18$	$B = 36$	$C = 42.$

The first player made 18 points, the second 36, and the third 42 points.

EXERCISE 5·5

1. The ratio of sand to gravel to cement by volume for concrete is 3 : 5 : 2. The construction worker needed 24 cubic yards of concrete. What volumes of sand, gravel and cement are required?

2. The ratio of the electricity used in a toaster to that of a coffee percolater to that of a fry pan is 3:2:10. A total of 450 watt-hours of electricity were used with these appliances. How many watt-hours did each use?

3. Three beneficiaries received an inheritance of $122,100 in the proportion 6 : 4 : 1. How much did each receive?

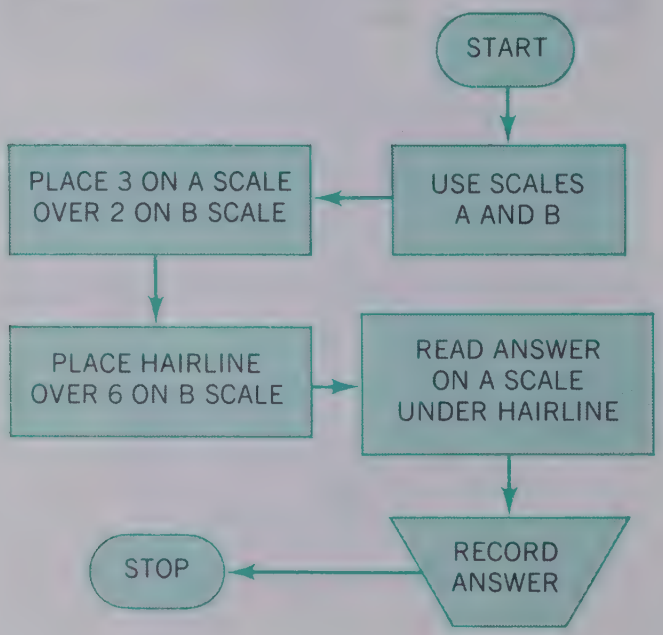
5·6 Using the Slide Rule

Follow the flow chart and diagram to learn how to use the slide rule to solve proportions.

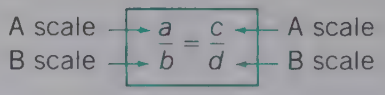
Example 1

Solve the proportion $\frac{3}{2} = \frac{n}{6}$.

Solution


START
USE SCALES A AND B
PLACE 3 ON A SCALE OVER 2 ON B SCALE
PLACE HAIRLINE OVER 6 ON B SCALE
READ ANSWER ON A SCALE UNDER HAIRLINE
RECORD ANSWER
STOP

This chart summarizes the use of the slide rule for all proportions.



EXERCISE 5·6

Solve each of the following using the slide rule. Round appropriately.

1. $\frac{3}{7} = \frac{n}{49}$ 2. $\frac{5}{13} = \frac{n}{52}$ 3. $\frac{n}{9} = \frac{63}{81}$

4. $\frac{n}{14} = \frac{66}{154}$ 5. $\frac{n}{18} = \frac{36}{64}$ 6. $\frac{7}{n} = \frac{54}{40}$

7. $\frac{27}{n} = \frac{60}{40}$ 8. $\frac{3}{4} = \frac{75}{n}$ 9. $\frac{24}{33} = \frac{54}{n}$

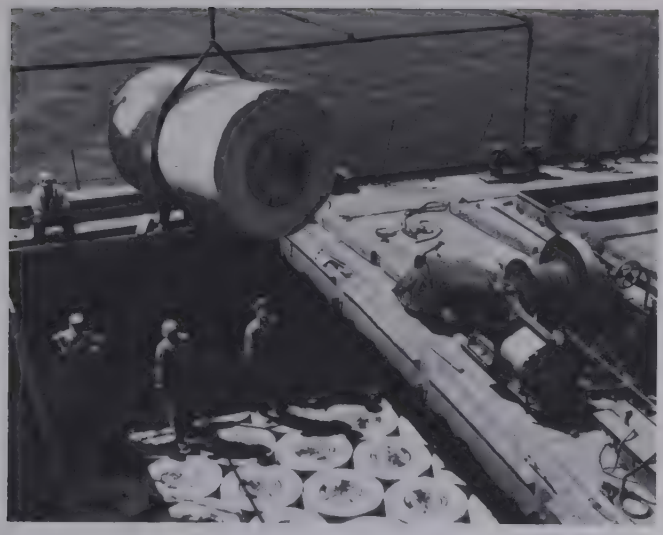
10. Write a flow chart for using the slide rule for question 4 and question 6.

5·7 Solving Problems Using Proportions

Example 1

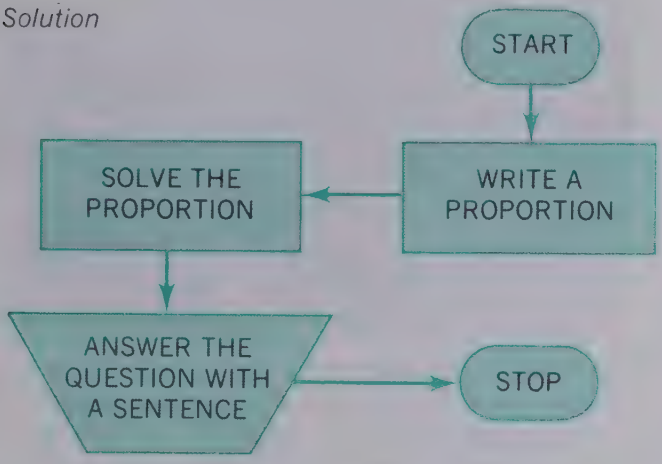
If it takes 35 trees to make 2 tons of newsprint, how many trees are necessary for 9,500,000 tons of newsprint (approximately Canada's annual production)?

Pulp logs, Smooth Rock Falls, Ontario.



Newsprint being loaded onto cargo ship on Saguenay River in central Quebec.

Solution



1. $\dfrac{\text{trees}}{\text{tons}} \longrightarrow \dfrac{35}{2} = \dfrac{n}{9{,}500{,}000} \longleftarrow \dfrac{\text{trees}}{\text{tons}}$

   Note! The second term must express the same ratio as the first term.

2. $$\frac{35}{2} = \frac{n}{9{,}500{,}000}$$
   $$n = 166{,}250{,}000$$

   Estimate:
   35 is about 20 times larger than 2. Therefore the answer is about 20 times 9,000,000. Hence, it is about 180,000,000.

   Is our answer correct?

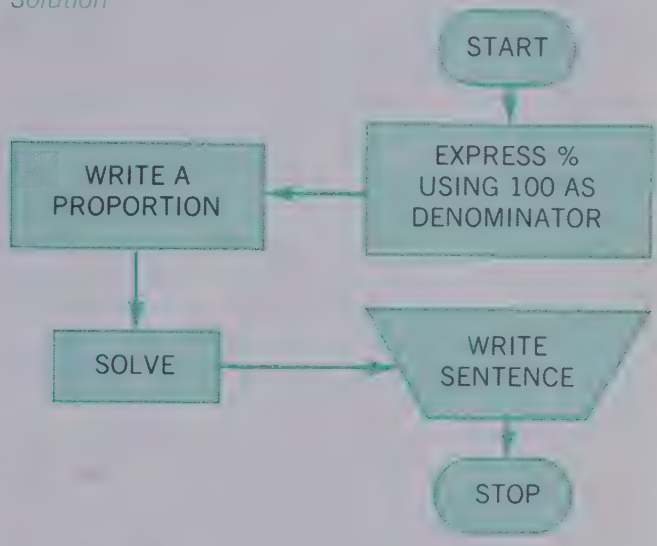
3. The number of trees is 166,250,000.

Percent questions can be worked by using proportion.

Example 2

Canada uses 76% of its paper production. How many tons of newsprint are used annually in Canada?

Solution



1. $76\% = \frac{76}{100}$

2. $\dfrac{\text{used}}{\text{produced}} \longrightarrow \dfrac{76}{100} = \dfrac{n}{9{,}500{,}000} \longleftarrow \dfrac{\text{used}}{\text{produced}}$

3. $n = 7{,}220{,}000$

4. 7,220,000 tons of newsprint are used annually in Canada.

   Can you estimate to check our answer?

EXERCISE 5·7

1. A car uses 6.3 gallons of gas to go 116 miles. At this rate how far will the car go on 25 gallons?

2. Draw a flow chart to show how to use your slide rule to find
   (a) what percent 15 is of 30,
   (b) 50 percent of 80,
   (c) what number 20 is 50 percent of.

3. Determine what percentage of your class smoke cigarettes regularly.

4. Determine what percentage of your class are girls.

5. Determine what percentage of your class drive motorbikes to school.

6. The rate of the rise to the run of a ski slope is 2 : 5. If the run is 1.3 miles, how high is the rise in feet?

7. The dimensions of a Golden Rectangle are approximately 1 : 1.6.
   Find the length of a Golden Rectangle
   (a) 42 centimetres wide
   (b) 48 centimetres wide
   (c) 72 centimetres wide
   (d) 22 centimetres wide.

8. The Sunshine Ski Area near Banff offers the skier a chairlift to the top of Mount Brewster. The run of this slope is $2\frac{1}{2}$ miles and the vertical rise is 1,850 feet. Express this as a ratio. How many feet of run is there for each yard of drop? Draw this slope in your book or on the chalkboard.

9. Down the back of the mountain into the Alberta side plunges the steepest ski slope of Canada, *Delirium Dive*. It drops 7.7 feet in every 10 feet of run. The run is about $1\frac{1}{3}$ miles. What is the vertical drop of Delirium Dive?

10. In the Toronto-Hamilton area in 1971 precipitation showed 90 parts of lead per billion. The maximum allowed in drinking water is 10 parts per billion. What percentage is the precipitation pollution level of the maximum allowed?

Is it clean snow?

11. In 1971 rainwater in the Guelph area of Ontario was found to contain cadmium (a pollutant) in the proportion of 18 : 1,000,000,000. How much cadmium is there in 5,000,000 pounds of rainwater? If water weighs 10 lb per gallon how many gallons contain 5 lb of cadmium?

12. The maximum amount of cadmium permitted in drinking water is 10 parts per billion. With this as a proportion what is the maximum amount of cadmium permitted in
   (a) 5,000,000 pounds of drinking water?
   (b) 4,500,000 gallons of water?

13. 18 gallons of gasoline produce approximately 23.3 grams of pollutants when burned in a car. How many grams of pollutants are produced by a car going 108 miles if the car gets 16 miles per gallon?

14. In Germany garbage and rubbish are burned and the heat utilized to generate steam to turn turbines to produce electricity. 700 tons of rubbish burned a day produces enough light and heat for 40,000 people.

    (a) If each person produces 5 pounds of garbage a day what percent of the required electricity can be produced by the garbage from 40,000 people?

    (b) How many pounds of garbage are required to light and heat a city of 345,000?

15. Find the missing numbers.

    6 is ■ % of 30
    45 is ■ % of 135
    175 is ■ % of 180

    35% of 80 = ■
    20% of 74 = ■
    125% of 64 = ■

    40 is 60% of ■
    32 is 80% of ■
    2.5 is 20% of ■

16. It is predicted that the number of cars in Halifax in the year 2000 will be 4 times the number in 1970. The number in 1970 was 2.5 times the number in 1952. In 1970 there were 0.75 tons of deadly sulphur dioxide from cars polluting the air daily. At that rate how many pounds of sulphur dioxide will be put into the air daily in 2000?

17. In 1964 there were about 2.6 persons for every car in Canada. If the population of Canada in 1964 was 19,560,000 how many cars were there in Canada that year?

18. If each car produced 500 kilograms of pollutant annually how many kilograms were discharged into the air in 1964?

19. In 1970 the population was 21,324,000. Based on the information in questions 17 and 18 how many cars were in Canada in 1970? How much pollutant was discharged into the air by cars?

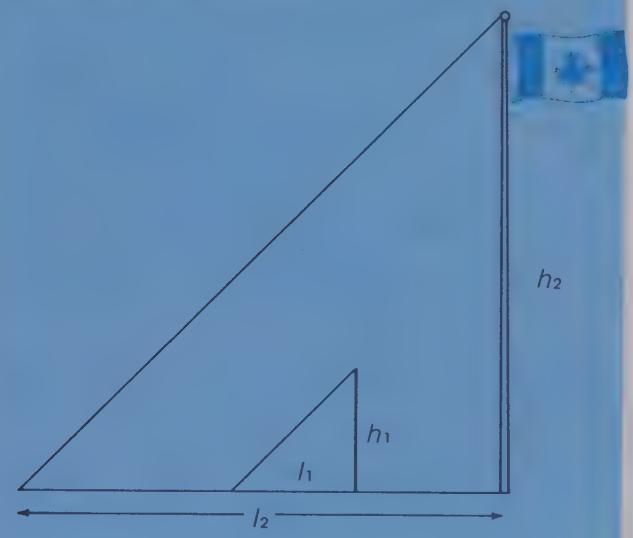
5·8 Similar Triangles

Projects

Project 1

On a sunny day measure the length of the shadow cast by a flagpole. Stand a yardstick vertically and measure the length of its shadow. Can you find the height of the flagpole? Read on *only if* you need help.

Calculate the height of the flagpole by using the following proportions for similar triangles.



$$\frac{\text{length of shadow of yardstick}}{\text{height of yardstick}} = \frac{\text{length of shadow of flagpole}}{\text{height of flagpole}}$$

$$\frac{l_1}{h_1} = \frac{l_2}{h_2}$$

Project 2

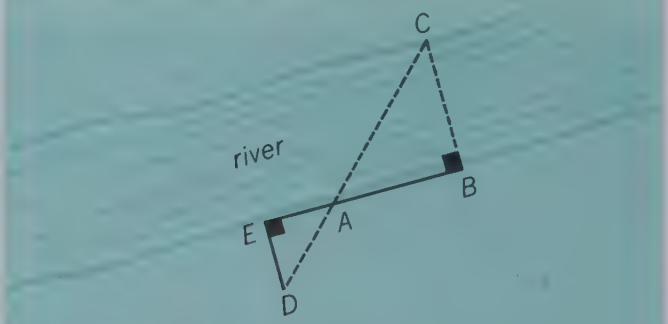
Find the height of a building assigned by your teacher.

Project 3

Find the height of a telephone pole, tree or lightpost assigned by your teacher.

Project 4

The diagram below illustrates a method of calculating the distance across a river.



(a) Measure some distance $\overline{BA}$.
(b) Measure some distance $\overline{EA}$.
(c) Measure some distance $\overline{ED}$ so that D, A and C are in a straight line.
(d) Use the proportion for similar triangles.

$$\frac{m(\overline{ED})}{m(\overline{CB})} = \frac{m(\overline{EA})}{m(\overline{AB})}$$

Questions:

1. Explain the formula in (d).
2. Use this method to calculate the width of the street in front of your school.
3. If it is safe, make a direct measurement of the width of the street.
4. Calculate the percentage error using

$$\text{percent error} = \frac{\text{error}}{\text{true measure}}.$$

5. Use this method to find the width of some area assigned by your teacher.

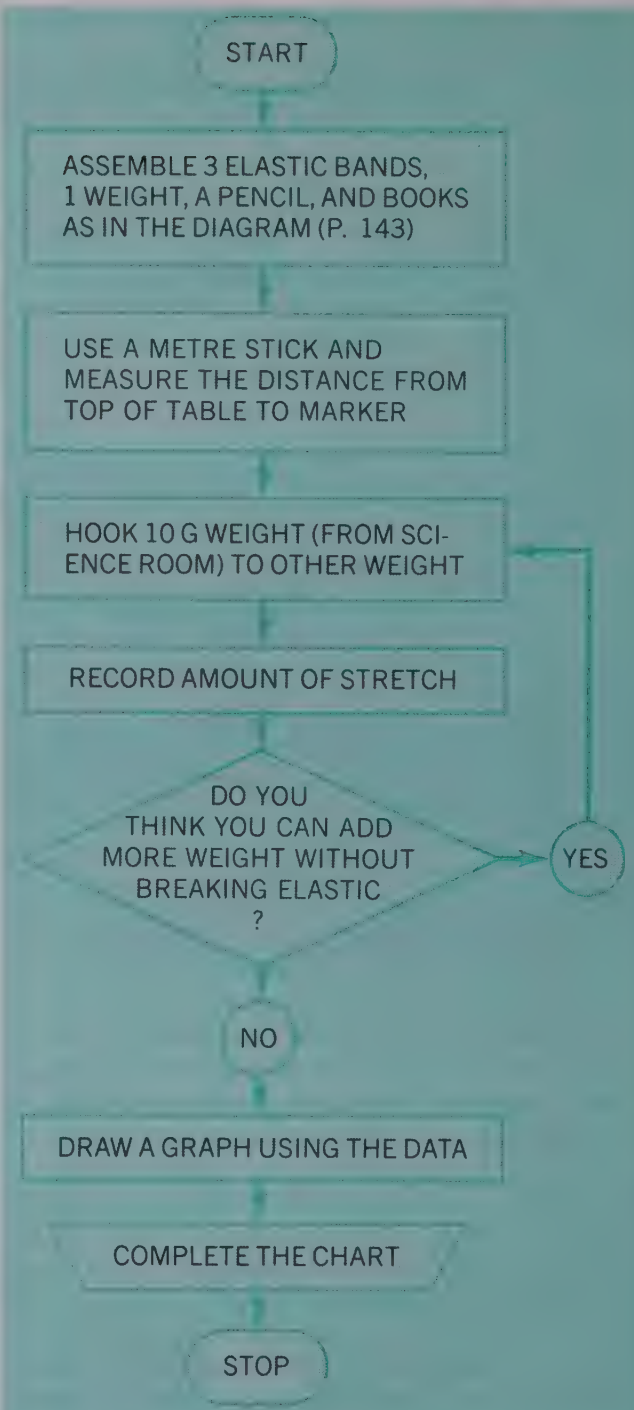
6·4 Direct Variation

Projects

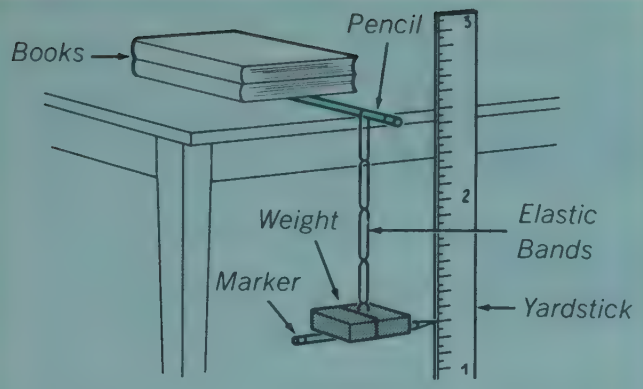
Project 5

Follow the flow chart on this page and answer the following questions.

1. Describe the type of graph obtained.
2. Each time 10 grams are added what happens to the amount of stretch?


START
ASSEMBLE 3 ELASTIC BANDS, 1 WEIGHT, A PENCIL, AND BOOKS AS IN THE DIAGRAM (P. 143)
USE A METRE STICK AND MEASURE THE DISTANCE FROM TOP OF TABLE TO MARKER
HOOK 10 G WEIGHT (FROM SCIENCE ROOM) TO OTHER WEIGHT
RECORD AMOUNT OF STRETCH
DO YOU THINK YOU CAN ADD MORE WEIGHT WITHOUT BREAKING ELASTIC ?
YES
NO
DRAW A GRAPH USING THE DATA
COMPLETE THE CHART
STOP

Weight in grams *(W)*	10	20	30	40
Stretch *(S)*				
Change in stretch				



3. Express the ratios $\frac{S}{W}$ in lowest terms for each bit of data. What do you notice? Complete these statements:

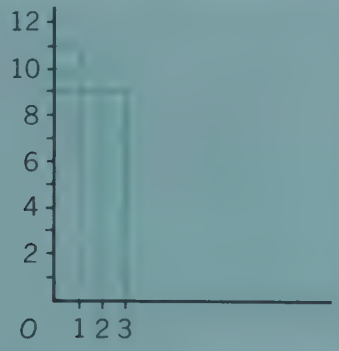
$$\frac{S}{W} = \blacksquare$$
$$S = \blacksquare \times W$$

This is an example of *direct variation.*

Project 6

How many different rectangles with a perimeter of 24 centimetres can you make if the length and width are to be integers? Make two rectangles of each size. Each size should be from different colored paper. Arrange them on a graph as indicated – each lying over top of the previous ones. Draw a curve to touch all outer corners of the rectangles.

Discuss the graph. What can you say about it? Each time you increase l by 1 what happens to w?



(a) What do you notice when replacements are made for l and w in $24 = 2(l + w)$?

5·10 Inverse Variation

Repeat Project 6 but make rectangles with an *area* of 24 square centimetres. Then answer this question instead of (a) in Project 6.

(b) What do you notice about $l \times w$ in each case? This is an example of an

Using the Library

REVIEW EXERCISE 5

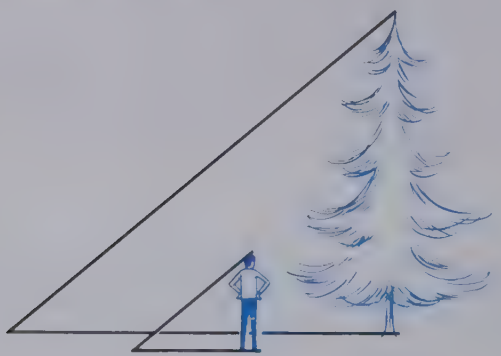
1. Determine whether or not each pair of ratios are equivalent.
   (a) $\frac{2}{3}, \frac{8}{12}$ (b) $\frac{5}{7}, \frac{80}{112}$
   (c) $3:4.2, 6.75:9.45$ (d) $7:1.2, 66:1.11$

2. Find the ratio of the area of a square of 2-inch sides to that of a square of 4-inch sides.

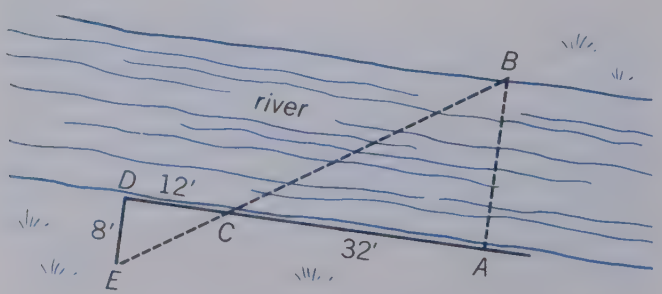
3. Find the ratio of the volume of a cube of 2-inch sides to that of a cube of 4-inch sides.

4. Three partners in a company share the dividends of $345,000 on the basis of the shares they own. The ratio of the shares owned by the three is $4:3:1$. How much does each receive?

5. A tree casts a shadow 84 feet long. At the same time a person 5 feet tall casts a shadow 4.3 feet. How tall is the tree?



6. Find the distance AB given the measures on the diagram.



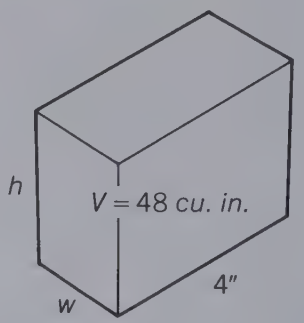
7. 1 mile = 1.6 kilometres.
   Complete the table to show the equivalents.

miles per hour	10	20	30	40	50	60	70	80
kilometres per hour								

   Show the information by a line graph.

8. If 120 feet of lumber cost $55, what would be the cost of 5,280 feet of lumber?

9. The volume of a box with fixed length of 4 inches is 48 cubic inches. What are the possible widths and heights?



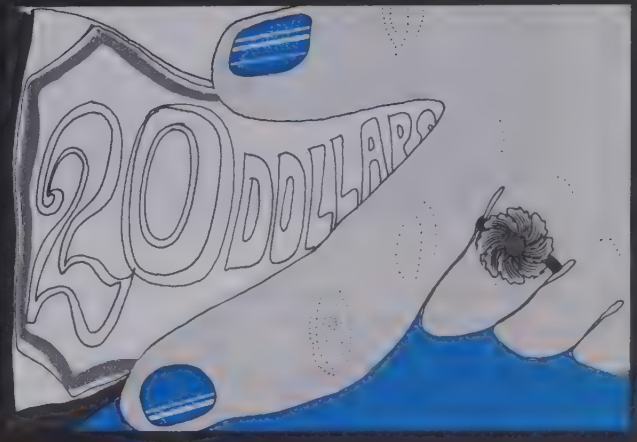
Hint! $V = l\,w\,h$
$48 = 4 \times w \times h$
$12 = w \times h$

w								
h								

Draw a line graph to show the information.

What vocation do you plan to follow? Make up a question using proportions related to your intended field of work or related to the pictures shown.

Chapter Six
PAYROLL



6·1 Getting a Job

Most students in your class will be employed at some time. Each will be paid. Each will have certain deductions made from his gross earnings. Each will want to know how to determine whether or not the deductions and calculations have been made properly. Hence this chapter will be of interest to you.

A number of students may have plans to work in offices after leaving high school. Many of these students will be involved in some of the payroll departments. Hence, for these students, this chapter should be of particular interest.

When an employer has a job vacancy he usually advertises the position. Normally the applicant is asked to apply in writing stating those particulars the employer would find relevant to the position.

Do questions 1 and 2 in Exercise 6·1.

When an applicant is accepted he is asked to supply information for a personal record card. This includes age, birth date, address, phone number, citizenship, married status, next of kin, previous experience and training. The applicant is require to complete a TD-1 income tax deduction declaration form. On the basis of the information on this form the payroll department makes the proper income tax deductions.

The County of Mountain View No. 17, with offices in Didsbury, Alberta, invites applications for the position of **STENOGRAPHER**

Please forward applications to the undersigned, giving details of experience, availability, marital status, typing speed, salary expected, two employment references, two character references. Shorthand not essential but desirable. Usual employee benefits. Excellent working conditions.

REQUIRED Immediately for year round employment, certified welders and experienced trailer mechanics. Top wages, comprehensive medical plan, good working conditions, liberal fringe benefits. Apply in person

SALES AND MERCHANDISING REPRESENTATIVE

Applicant must be aggressive and dependable. For this salaried, permanent position. Travelling expenses and automobile supplied. Employee benefits include: group achievement fund, healthy plan, group insurance, pension. Five day week. Our employees are aware of this position.

MODELS

or model types, 19-25, will be offered $96 a week to train as counsellor for an exclusive European skin care organization. This is not door-to-door cosmetic selling, and only those willing to counsel the beauty needs of our customers need apply. Experience not essential.

105 Help Wanted

"MORE JOBS IN MORE PLACES"

LEGAL SECRETARY: $400. to $500 — Be right hand Sr. partner in busy well known law firm. Utilize your top secretarial skills and legal background in all phases of law, especially mortgages and litigation.

PAYROLL — $350+. To handle all salaries payroll on N.C.R. 3300, distributions, cheques, and journal recording. Must be able to work on own, and have a pleasant telephone manner to join this excellent Oil Company in two weeks.

LARGE CANADIAN COMPANY

Have openings in Vancouver for 4 young men, aged 17 - 19. Applicant must be neat in appearance, free to travel and be interested in a career with above average earnings. Apply

PLANNING ASSISTANT

required

to perform duties related to facility planning. Successful applicant must be able to prepare charts, maps and graphs; compile statistical data related to schools, etc. Performs other routine clerical duties as required.

QUALIFICATIONS:

Standard High School completion. Undertaking of and interest in statistical and map work. Basic typing and drafting skill. Good knowledge of City of Calgary. 1 to 3 years' related experience. Personal suitability.

SALARY: $423.50 — $535.00 per month

APPLY IN PERSON:

EXERCISE 6·1

1. Obtain a local paper (or the daily paper from a large metropolitan area). Study the classified ads and the Career Opportunities and Help Wanted sections. What information do the ads give about the job and type of individual wanted?

2. Select, from page 146, one of the opportunities that appeals to you and write an application for the job (or select a job from the newspaper want ads in question 1).

Obtain from the local Income Tax Office copies of the TD-1 form.

3. Complete a TD-1 return for yourself.

4. Complete a TD-1 return for each of the following:
   (a) married man, John D. Ford, Born 23 July, 1950, claiming for two children of 4 and 2 years, spouse earning $650 annually.
   (b) married man, Fred Jennings, Born 12 August, 1952. Married for two months. Previous earnings of wife $450. Expected earnings of wife while married is $800.

6·2 Methods of Payment

Now that you have selected a job you will be interested in your first pay cheque. There are a number of pay arrangements: weekly (52), bi-weekly (26), twice monthly (24), monthly (12). The numbers in brackets indicate the number of payments in a year.

Salary

Most office workers are paid a *salary*. This amount is fixed and is stated in terms of a dollar payment per pay period or per year. The salary may be increased periodically by *raises* or *increments*. Raises are usually thought of as an increase in pay for knowing and doing the job better and for efficiency and higher performance as well as increased responsibility. Sometimes a raise is considered as *merit pay*. When one's effectiveness is difficult to assess and when it is assumed that one's performance increases with time, increments are sometimes paid on a regular basis.

Hourly Wage

Many plant workers and laborers work by the hour. The longer one works the more he achieves (normally) and hence the more he is paid. Carpenters, mechanics, production line workers in plants, electricians, repair men, part time waitresses and clerks are a few that may be in this category. The number of hours one may work at regular pay is usually agreed upon by the employer and employee (or by agreement between employer and union officials). Any *overtime* is usually paid on an increased hourly rate.

Example 1

Mr. C. U. Piper, a plumber, charges $8.50 an hour for any work between 8.00 a.m. and 5.00 p.m. and *time-and-a-half* for overtime. A house owner phones for the plumber at 2.30 p.m. on an emergency basis—his basement is flooding. The plumber works until 6.45 p.m. to repair the ruptured water lines. Parts cost $33.30. What is the plumber's bill?

Solution

Regular pay from 2.30 to 5.00 p.m. (includes travelling time):

$$2\tfrac{1}{2} \times 8.50 = \$21.25$$

Time-and-half overtime from 5.00 to 6.45 p.m. rate:

$$1\tfrac{1}{2} \times 8.50 = \$12.75 \text{ per hour}$$

Amount for $1\frac{3}{4}$ hours:

$$1\tfrac{3}{4} \times 12.75 = \$22.31$$

C. U. Piper,
Plumber,
3152 Union Street

ALL ACCOUNTS DUE UPON RECEIPT May 4,1971

Hrs.	Rate	Amount	
2½	$8.50	21	25
1¾	$8.50 x 1½ = $12.75	22	31
Labor		43	56
Parts	Pipes, Connections, Valve	33	30
	TOTAL	76	86

People who work in plants or factories usually *punch the clock* on arrival and on leaving. This means they insert their time card into a special clock which stamps the time on the card.

"SETTLE AN ARGUMENT, WILL YOU? SHE SAYS SHE NEEDS A MINK BIKINI!"

NAME: MICHAEL B. BRASH
PAYROLL NO.: 11144
SOCIAL INSURANCE NO.: 329-671-721
WEEK ENDING: May, 15, 1971
DEPARTMENT: Finishing

DAY	A.M.		P.M.		OVERTIME		TOTAL	
	IN	OUT	IN	OUT	IN	OUT	REGULAR	OVERTIME
Mon	7:59	12:02	1:00	5:01			8	
Tue	7:55	12:02	12:55	5:00	5:00	6:30	8	$1\frac{1}{2}$
Wed	**8:05**	12:03	12:56	5:01			$7\frac{3}{4}$	
Thurs	7:55	12:02	12:58	**4:56**			$7\frac{3}{4}$	
Fri	7:55	**11:40**	12:56	5:00	5:00	6:00	$7\frac{1}{2}$	1
Sat					8:00	10:00		2
Sun								

	HOURS	RATE	EARNINGS
Regular	39	$3.22\frac{1}{2}$	$125.77
Overtime ($1\frac{1}{2}$)	$4\frac{1}{2}$	$4.83\frac{3}{4}$	21.75
Gross Pay			$147.52

There are several items to be noted on this time card. Notice those times in heavy print. In each case Michael Brash punched in *late* or punched out *early*. If he is late 1 to 15 minutes he is deducted 15 minutes; if he is late 16-30 minutes he is deducted 30 minutes. The same principle applies to leaving early. There is no compensation for arriving early or leaving late.

Also notice that Mr. Brash is on a 40 hour week. The time he works on Saturday or Sunday is considered overtime.

The hourly wage earner is not on a fixed salary. If he is absent because of illness, he does not get paid. Very often hourly wage earners are protected from loss of pay due to illness by labour union contracts which provide for *sick-pay*.

EXERCISE 6·2

1. A secretary's salary is $135.60 per pay period. What is the annual salary if the secretary is paid:
   (a) weekly
   (b) bi-weekly
   (c) semi-monthly
   (d) monthly

2. Four office workers are each paid $8,500 per annum.
   (a) What is the gross pay per period of each if the first is paid weekly, the second is paid bi-weekly, the third, semi-monthly and the last, monthly?
   (b) Which person would you prefer to be? Why?
   (c) Which do you think finds it the easiest to stay on a budget? Why?

3. Mr. R. U. Coloure, a television repair man, charges $6.50 an hour and time-and-a-half overtime. He is called to repair a set at 3.00 p.m. the day before the Grey Cup. He works until 7.00 p.m. The parts cost $28.45. Write up a bill (as in Example 1) for Mr. R. U. Coloure.

4. Prepare time cards for each employee of the Western Plywood Company for the week ending June 28th. Regular hours are 8.00 a.m. to 12.00 noon and 1.00 p.m. to 5.00 p.m. Monday through Friday. Time-and-a-half overtime is paid.
   (a) Paul Newman, Gluing Department, Payroll No. 44111, Social Insurance No. 321-922-333. Rate $2.75 per hour.

Monday:	7.55 – 12.01;	12.55 – 5.03
Tuesday:	8.05 – 12.01;	12.58 – 5.05
Wednesday:	7.55 – 11.50;	12.55 – 5.05
Thursday:	7.55 – 12.01;	1.00 – 5.00
		5.00 – 6.30
Friday:	7.50 – 12.01;	1.10 – 5.00
Saturday:	8.00 – 12.01	

   (b) Henry Flattop, Trimming Department, Payroll No. 44113, Social Insurance No. 562-721-861. Rate $3.05 per hour.

Monday:	7.50 – 12.00;	1.00 – 4.50
Tuesday:	7.55 – 12.02;	12.55 – 5.10
Wednesday:	7.50 – 12.05;	12.56 – 5.00
		5.00 – 6.30
Thursday:	8.10 – 12.00;	12.58 – 5.00
Friday:	7.57 – 12.01;	1.20 – 5.00
Saturday:		12.57 – 5.05

6·3 Methods of Payment Continued

Piecework

Many employers wish to pay their employees for what they have done. Then if the employee works slowly he doesn't get paid as much as the employee who works more quickly. Under the hourly wage scheme, there sometimes is the tendency to punch-in and punch-out on time and do little else. Under the piecework scheme the concern is with actual production.

Under the piecework basis, a worker is paid a stated number of dollars (or cents) per unit produced. The examples and exercises will illustrate this pay method in operation.

Example 1

The Western Prairie Clothes Manufacturer operates its sewing department on a piecework basis. Women's dresses are sewn in three separate stages by women doing specific work. The rate for Stage 1 is 22¢ per unit completed. Calculate the weekly wage of a worker who completes 75 units per day.

Payment for 75 units is $75 \times \$ \ .22 = \$16.50$
Payment for 5 days at \$16.50 $= \$82.50$
The weekly wage is \$82.50.

Bonus Piecework

In order to encourage employees to work at a higher level, an employer will often pay a bonus for each unit produced over a certain number.

Example 2

A lumberjack is paid daily at the rate of 18¢ for each tree felled and marked up to 100 trees. Then he receives 24¢ for each tree he falls over 100. Paul Bunyon felled, on one day, 134 trees. What was his wage?

Solution

Earnings on 100 trees is $100 \times \$ \ .18 = \$18.00$
Earnings on next 34 trees is $34 \times \$ \ .24 = \$8.16$
Total earnings = \$26.16

Differential Piecework

While bonus piecework encourages each employee to work at full capacity, a differential piecework scheme is even more encouraging. In this scheme the employee receives a certain amount per unit produced up to a certain number. If he produces over that number he gets a higher rate per unit for all units.

Example 3

An electrical appliances manufacturer, "The Ohms Bell Company", pays its employees according to the following schedule:

PRODUCTION	DIFFERENTIAL RATE PER UNIT
1 – 50	16¢
51 – 75	20¢
76 – 100	24¢
Over 100	28¢

Calculate the wage of the employee who produces 83 units.

Solution

Since 83 is in the range 76 – 100 the employee will receive 24¢ per unit.
Therefore 83 × $ .24 = $19.92
The wage is $19.92.

Other Methods

Some companies pay their employees on a commission basis. This method will be discussed in a later book. Other methods include
(a) bonus above regular wage,
(b) benefit on profit sharing.

If a company has had a good year it will sometimes pass on to the employees a lump payment to supplement their regular wage. Other companies may encourage their respective employees to produce more for the benefit of the company by giving its employees shares or stocks in the company or a share of the profit.

EXERCISE 6·3

1. Calculate the weekly wage of an employee of Stage 1 of the Western Prairie Clothes Manufacturer (see Example 1) if he produces 62 units per day.

2. Workers do not always produce the same number of units per day. Calculate the weekly wage of the employee who completes the number of units shown here (22¢ per unit).

Monday	– 58 units	Tuesday	– 69 units
Wednesday	– 71 units	Thursday	– 76 units
Friday	– 64 units		

3. A worker is paid daily. He is paid 8¢ per unit on the first 200 units, 10¢ per unit on all the rest. What is his wage for the week if he produces as shown:

Monday	– 245	Tuesday	– 190
Wednesday	– 232	Thursday	– 285
Friday	– 180		

4. A wheat combine operator is using his combine working at the rate of 90¢ per acre. In four days he combines 337 acres. What is his wage?

5. An employer pays $8.00 for the daily quota of 150 units. He then pays 75¢ for each unit over the quota. One employee had this production record. Calculate the weekly wage.

Monday	– 164 units	Tuesday	– 155 units
Wednesday	– 185 units	Thursday	– 190 units
Friday	– 158 units		

6. Use the differential rate schedule in Example 3. Calculate the wage of the individual who produces:

(a) 68 units	(b) 112 units
(c) 42 units	(d) 100 units
(e) 79 units	(f) 76 units

7. A publicity company pays occasional workers to address envelopes. The rates are:

Production	*Rate Per Unit*
1 – 100	6¢
101 – 200	8¢
201 – 300	10¢
Over 300	12¢

Calculate the wage of the employee who addresses:

(a) 90 envelopes	(b) 278 envelopes
(c) 146 envelopes	(d) 315 envelopes

Project

Project

Make a survey of your area and list a variety of employers. Set up a phoning system in which each member of your class phones one or two of the employers. Enquire as to the pay methods of each firm. Record the data in some suitable form. Discuss why you think each employer uses the pay system he does.

Don't forget to prepare what you are going to say to each employer. Explain why you want the information.

6·4 Getting Paid

Now that we have discussed getting a job and the basis upon which one is paid, we shall now discuss the actual statement of earnings that very often accompanies the pay cheque or pay envelope.

Methods of payment vary a great deal but we shall generalize considerably and discuss two methods in particular with those factors common to most methods of payment.

Cash

Some firms pay their employees in cash placed in a sealed envelope. The envelope has a statement of gross earnings, deductions, net payment and the breakdown of the cash contained. While this practice is on the decrease, many employees still favour it over receiving a cheque.

Example 1

Prepare a pay envelope for Michael B. Brash (see page 148, Section 6.2).

Deductions are:

Income Tax	$29.75
Unemployment Insurance	$ 1.40
Canada Pension	$ 2.45
Health Plan	$ 1.12

Solution

REGULAR:	*39 hrs. @ 3.22\frac{1}{2}$*	*$125.77*
OVERTIME:	*$4\frac{1}{2}$ hrs. @ 4.83\frac{3}{4}$*	*21.75*
	TOTAL	*$147.52*

NAME:	*Michael B. Brash*
PAYROLL NO.	*11144*
SOCIAL INSURANCE NO.	*329-671-721*
WEEK ENDING	*May 15, 1971*
DEPARTMENT	*Finishing*

GROSS EARNINGS		*$147.52*
DEDUCTIONS		
INCOME TAX	*$29.75*	
UNEMPLOYMENT INS.	*1.40*	
CANADA PENSION	*2.45*	
HEALTH PLAN	*1.12*	*34.72*
NET EARNINGS		*$112.80*

5 × $20	*100.*	
1 × $10	*10.*	
× $ 5		
1 × $ 2	*2.*	
× $ 1		
80 COINS	*.80*	
		$112.80

One of the most common arrangements for paying employees is by cheque. The cheque normally has a stub or accompanying statement showing particulars.

National Bank — May 15, 1972

PAY TO THE ORDER OF ..... Michael B. Brash $112.80

One Hundred Twelve ..... 80/100 DOLLARS

J. E. JOHSON
THE CEDAR PLYWOOD COMPANY *J. E. Johson*

STATEMENT OF EARNINGS

	Hours	Rate	Gross
REGULAR	39	$3.22½	$125.77
OVERTIME	4½	$4.83¾	21.75
			$147.52

NAME	M. B. Brash
PAYROLL NO	11144
SOC. INS. NO	327-671-721
WEEK ENDING	May 15, 1971
DEPARTMENT	Finishing

Deductions

Unemployment Insurance	$1.40	*Health Plan*	$ 1.12
Canada Pension	$2.45	*Income Tax*	$29.75

NET $112.80

The employer is responsible for paying to the government or other firms (health and medical insurance) on behalf of the employees. Therefore it is necessary for the employer to keep detailed accounts of earnings, deductions and payments.

EXERCISE 6·4

1. Discuss the reasons an employee would favor receiving cash rather than a cheque.

2. Discuss the reasons an employee would favor receiving a cheque rather than cash.

3. Prepare a pay envelope for Georgy Brieson, payroll number 23477; Social Insurance No. 456-721-771; week ending March 3, 1971; gross earnings $121.75; Income tax $22.50; Unemployment insurance $1.40; Canada Pension $1.98; Health Plan $1.15.

4. Alice Brooks received a cheque and statement for the week ending May 28, 1971. As any employee should do, she checked the calculations. If you were Alice Brooks what would you do?

STATEMENT OF EARNINGS

	Hours	Rate	Gross
REGULAR	38	$2.81	$96.78
OVERTIME	8½	$4.20½	$35.74
			$132.52

NAME	A. Brooks
PAYROLL NO	772-31
SOC. INS. NO	561-771 325
WEEK ENDING	July 30, 1972
DEPARTMENT	Paint

Deductions

Unemployment Insurance	$1.40	*Health Plan*	$1.05
Canada Pension	$1.71	*Income Tax*	$18.75

NET $99.61

6·5 Deductions

Let us review briefly how the employer or the accountant or payroll clerk determines how much to deduct from each employee's weekly earnings. We will discuss weekly pay. Bi-weekly, semi-annual or monthly pay is handled in a similar manner.

Income Tax

Employers are required under the Canadian Income Tax Law to deduct the proper amount of personal income tax from their employees' earnings. This amount is determined from the TD-1 Tax Deduction Declaration Form and from an Income Tax Deduction Table. To assist the employer the government supplies detailed tax deduction tables. A portion of a table is shown on page 154.

TABLE 136

WEEKLY TAX DEDUCTIONS
Basis—52 Pay Periods per Year

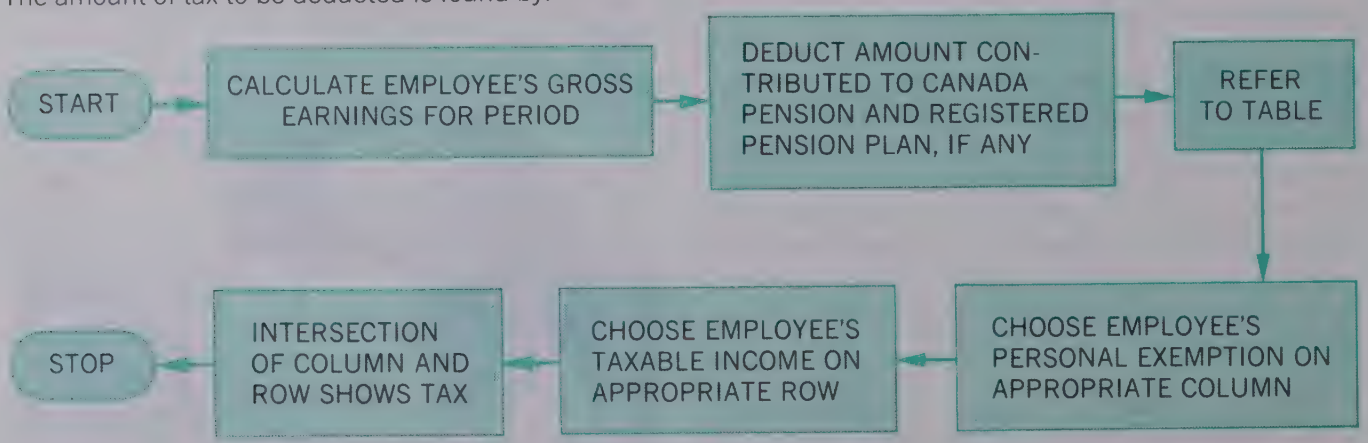
DÉDUCTIONS D'IMPÔT PAR SEMAINE
Base—52 périodes de paie par année

WEEKLY PAY Use appropriate bracket — PAIE PAR SEMAINE Utilisez le palier approprié	IF THE TOTAL OF EXEMPTIONS IS — SI LE TOTAL DES EXEMPTIONS EST DE														
	$1,000-$1,249	$1,250-$1,499	$1,500-$1,749	$1,750-$1,999	$2,000-$2,249	$2,250-$2,499	$2,500-$2,749	$2,750-$2,999	$3,000-$3,249	$3,250-$3,499	$3,500-$3,749	$3,750-$3,999	$4,000-$4,249	$4,250-$4,499	$4,500-$4,750
	DEDUCT FROM EACH PAY — DÉDUISEZ SUR CHAQUE PAIE														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Under-Moins de $23.00	.00														
$ 23.00 - 23.99	.30														
24.00 - 24.99	.45														
25.00 - 25.99	.60														
56.00 - 56.99	6.25	5.25	4.25	3.25	2.40	1.65	.90	.20							
57.00 - 57.99	6.45	5.45	4.45	3.45	2.55	1.80	1.05	.35							
58.00 - 58.99	6.65	5.65	4.65	3.65	2.70	1.95	1.25	.50							
59.00 - 59.99	6.85	5.85	4.85	3.85	2.90	2.10	1.40	.65							
60.00 - 60.99	7.10	6.05	5.05	4.05	3.05	2.25	1.55	.80	.05						
61.00 - 61.99	7.35	6.30	5.25	4.25	3.30	2.45	1.70	.95	.20						
62.00 - 62.99	7.60	6.50	5.45	4.50	3.50	2.60	1.85	1.10	.35						
63.00 - 63.99	7.80	6.70	5.70	4.70	3.70	2.75	2.00	1.25	.50						
64.00 - 64.99	8.05	6.90	5.90	4.90	3.90	2.90	2.15	1.40	.65						
65.00 - 65.99	8.30	7.15	6.10	5.10	4.10	3.10	2.30	1.55	.85	.10					
81.00 - 81.99	12.25	11.05	9.85	8.70	7.50	6.45	5.45	4.45	3.45	2.55	1.80	1.05	.35		
82.00 - 82.99	12.50	11.30	10.10	8.95	7.75	6.65	5.65	4.65	3.65	2.70	1.95	1.20	.50		
83.00 - 83.99	12.80	11.55	10.35	9.20	8.00	6.85	5.85	4.85	3.85	2.90	2.10	1.40	.65		
84.00 - 84.99	13.05	11.80	10.60	9.45	8.25	7.10	6.05	5.05	4.05	3.05	2.25	1.55	.80	.05	
85.00 - 85.99	13.30	12.05	10.85	9.65	8.50	7.35	6.25	5.25	4.25	3.25	2.40	1.70	.95	.20	
86.00 - 86.99	13.60	12.30	11.10	9.90	8.75	7.55	6.50	5.45	4.45	3.50	2.55	1.85	1.10	.35	
87.00 - 87.99	13.85	12.55	11.35	10.15	9.00	7.80	6.70	5.70	4.70	3.70	2.75	2.00	1.25	.50	
88.00 - 88.99	14.10	12.85	11.55	10.40	9.25	8.05	6.90	5.90	4.90	3.90	2.90	2.15	1.40	.65	
89.00 - 89.99	14.40	13.10	11.85	10.65	9.45	8.30	7.15	6.10	5.10	4.10	3.10	2.30	1.55	.80	.10
90.00 - 91.99	14.65	13.35	12.10	10.90	9.70	8.55	7.35	6.30	5.30	4.30	3.30	2.45	1.70	.95	.25
$ 92.00 - $ 93.99	15.15	13.90	12.60	11.40	10.20	9.05	7.85	6.75	5.70	4.70	3.70	2.75	2.00	1.30	.55
94.00 - 95.99	15.70	14.45	13.15	11.90	10.70	9.50	8.35	7.20	6.15	5.15	4.15	3.15	2.35	1.60	.85
96.00 - 97.99	16.25	14.95	13.70	12.40	11.20	10.00	8.85	7.65	6.55	5.55	4.55	3.55	2.65	1.90	1.15
98.00 - 99.99	16.75	15.50	14.20	12.95	11.65	10.50	9.30	8.15	7.00	5.95	4.95	3.95	3.00	2.20	1.45
100.00 - 101.99	17.35	16.00	14.75	13.45	12.20	11.00	9.80	8.65	7.45	6.40	5.40	4.40	3.40	2.50	1.75
102.00 - 103.99	17.95	16.55	15.30	14.00	12.75	11.45	10.30	9.15	7.95	6.80	5.80	4.80	3.80	2.85	2.10
104.00 - 105.99	18.55	17.10	15.80	14.55	13.25	12.00	10.80	9.60	8.45	7.25	6.20	5.20	4.20	3.20	2.40
106.00 - 107.99	19.15	17.70	16.35	15.05	13.80	12.50	11.25	10.10	8.95	7.75	6.65	5.65	4.65	3.65	2.70
108.00 - 109.99	19.75	18.30	16.90	15.60	14.30	13.05	11.75	10.60	9.40	8.25	7.10	6.05	5.05	4.05	3.05
110.00 - 111.99	20.35	18.90	17.45	16.10	14.85	13.55	12.30	11.10	9.90	8.75	7.55	6.45	5.45	4.45	3.45
112.00 - 113.99	20.95	19.50	18.05	16.65	15.40	14.10	12.85	11.55	10.40	9.20	8.05	6.90	5.90	4.90	3.90
114.00 - 115.99	21.50	20.10	18.65	17.25	15.90	14.65	13.35	12.10	10.90	9.70	8.55	7.35	6.30	5.30	4.30
116.00 - 117.99	22.10	20.70	19.25	17.85	16.45	15.15	13.90	12.60	11.35	10.20	9.00	7.85	6.70	5.70	4.70
118.00 - 119.99	22.70	21.30	19.85	18.40	17.00	15.70	14.40	13.15	11.85	10.70	9.50	8.35	7.15	6.15	5.10

EXEMPTIONS OVER $4,750
Reduce the tax in Column 15 by 70c for each $250 (or part) of additional exemption.

EXEMPTIONS DÉPASSANT $4,750
Réduire l'impôt dans la Colonne 15 de 70c. pour chaque $250 (ou partie) d'exemption supplémentaire.

The amount of tax to be deducted is found by:


START
CALCULATE EMPLOYEE'S GROSS EARNINGS FOR PERIOD
DEDUCT AMOUNT CONTRIBUTED TO CANADA PENSION AND REGISTERED PENSION PLAN, IF ANY
REFER TO TABLE
CHOOSE EMPLOYEE'S PERSONAL EXEMPTION ON APPROPRIATE COLUMN
CHOOSE EMPLOYEE'S TAXABLE INCOME ON APPROPRIATE ROW
INTERSECTION OF COLUMN AND ROW SHOWS TAX
STOP

Do question 1 of Exercise 6·5.

Canada Pension Plan

Prior to the implementation of the Canada Pension Plan on July 1, 1966, many firms had their own retirement plans. The Canada Pension Plan is one operated by the Canadian Government for all employed Canadians. The chief reason for the implementation of the Canada Pension Plan, rather than the continuation of only the private plans, were:

(a) to provide a pension plan for all employees (with the exception of employees in Quebec which administers a similar plan independent of the Federal plan).

(b) private plans were not *portable*. That is, the contributions accumulated by an employee with one firm could not be transferred to another firm if the employee changed employers. The employee had to take a refund of a portion of his contributions plus interest but he had to start all over again with the new firm.

To calculate the amount of deductions use the table shown on this page. Only portions of the table are shown.

Many companies, unions and professions still maintain their private pension plans (modified slightly) in addition to the Canada Pension Plan. Employees of these organizations pay to both plans and on retirement will receive pensions from two sources.

CANADA PENSION PLAN
RÉGIME DE PENSIONS DU CANADA

WEEKLY PAY PERIOD
PÉRIODE HEBDOMADAIRE DE PAIE
$.00 — $171.26

Remuneration *Rémunération*	Deduct *Retenir*	Remuneration *Rémunération*	Deduct *Retenir*
21.27 — 21.81	.18	91.27 — 91.81	1.44
21.82 — 22.37	.19	91.82 — 92.37	1.45
22.38 — 22.92	.20	92.38 — 92.92	1.46
22.93 — 23.48	.21	92.93 — 93.48	1.47
23.49 — 24.03	.22	93.49 — 94.03	1.48
24.04 — 24.59	.23	94.04 — 94.59	1.49
24.60 — 25.14	.24	94.60 — 95.14	1.50
25.15 — 25.70	.25	95.15 — 95.70	1.51
25.71 — 26.26	.26	95.71 — 96.26	1.52
61.27 — 61.81	.90	96.27 — 96.81	1.53
61.82 — 62.37	.91	96.82 — 97.37	1.54
62.38 — 62.92	.92	97.38 — 97.92	1.55
62.93 — 63.48	.93	97.93 — 98.48	1.56
63.49 — 64.03	.94	98.49 — 99.03	1.57
64.04 — 64.59	.95	99.04 — 99.59	1.58
64.60 — 65.14	.96	99.60 — 100.14	1.59
65.15 — 65.70	.97	100.15 — 100.70	1.60
65.71 — 66.26	.98	100.71 — 101.26	1.61
81.27 — 81.81	1.26	111.27 — 111.81	1.80
81.82 — 82.37	1.27	111.82 — 112.37	1.81
82.38 — 82.92	1.28	112.38 — 112.92	1.82
82.93 — 83.48	1.29	112.93 — 113.48	1.83
83.49 — 84.03	1.30	113.49 — 114.03	1.84
84.04 — 84.59	1.31	114.04 — 114.59	1.85
84.60 — 85.14	1.32	114.60 — 115.14	1.86
85.15 — 85.70	1.33	115.15 — 115.70	1.87
85.71 — 86.26	1.34	115.71 — 116.26	1.88
86.27 — 86.81	1.35	116.27 — 116.81	1.89
86.82 — 87.37	1.36	116.82 — 117.37	1.90
87.38 — 87.92	1.37	117.38 — 117.92	1.91
87.93 — 88.48	1.38	117.93 — 118.48	1.92
88.49 — 89.03	1.39	118.49 — 119.03	1.93
89.04 — 89.59	1.40	119.04 — 119.59	1.94
89.60 — 90.14	1.41	119.60 — 120.14	1.95
90.15 — 90.70	1.42	120.15 — 120.70	1.96
90.71 — 91.26	1.43	120.71 — 121.26	1.97

Unemployment Insurance

Government regulations require the employer (with some exceptions) to deduct from the employee's earnings an amount (premium) for unemployment insurance. Like any insurance plan a person only collects under certain conditions:

(a) a person must be unemployed but willing and able to work.

(b) a person must have been employed at least 30 weeks (not necessarily consecutive) in the two years prior to unemployment.

The unemployed cause a great deal of concern to governments. What can governments do to solve unemployment?

Contributions are made jointly by the employer and employee. The table shows the total contributions and the contributions by the employee. The employer must pay the balance.

UNEMPLOYMENT INSURANCE COMMISSION

TABLE OF CONTRIBUTIONS AND DEDUCTIONS FROM WAGES

SYMBOL	Gross Earnings during ONE WEEK OR LESS	1 Employee Deduction	Employer Contribution	Total Weekly Contribution
1/2 W or 1I	Up to $19.99	$ .10	$ .10	$ .20(a)
	$ 20.00 to $ 29.99	.20	.20	.40
	30.00 to 39.99	.35	.35	.70
	40.00 to 49.99	.50	.50	1.00
1W OR 12	50.00 to 59.99	.65	.65	1.30
	60.00 to 69.99	.80	.80	1.60
	70.00 to 79.99	.95	.95	1.90
	80.00 to 89.99	1.10	1.10	2.20
	90.00 to 99.99	1.25	1.25	2.50
	100.00 and over	1.40	1.40	2.80

A new unemployment insurance act came into force on June 27, 1971.

Medical and Hospitalization Plan (Health Plan)

All provinces now have a compulsory medical plan. The plans vary from province to province but basically they provide the insured with free medical care and hospitalization. Most plans pay only basic rates to both doctors and hospitals. If the patient wishes to have more than basic provisions then he must pay extra. The premiums are often deducted from the employee's earnings and forwarded directly by the employer to the provincial government.

Other Deductions

Many firms agree to make deductions from the employee's earnings for various other purposes. These deductions are forwarded by the employer to the appropriate places. Some of the deductions are for:

(a) Union dues – Employees who belong to a union must pay fees to support the union. The employee and union reach an agreement on the amount of the dues.

(b) Group insurance – The insurance may be life, further medical or hospital insurance, or disability insurance. The management forwards these deductions to the appropriate insurance company. Sometimes there is an agreement that the employer pays a part of these insurance premiums while the employee pays the balance through deductions from his earnings.

EXERCISE 6·5

Use the appropriate table of deductions.

1. Calculate the income tax given the following information in each case. Listed, in order, are gross weekly earnings, Unemployment Insurance, Canada Pension, registered pension and number of exemptions.

Miss Jill Devitt
$85.10
1.10
1.32
2.30
1

Victor Kozar
$116.00
1.40
1.89
3.10
2

Richard Allan
$90.50
1.25
1.42
2.40
3

Mrs. C. Daniels
$65.00
.80
.97
1.12
9

	Weekly Gross Income	Income Tax	Unemployment Insurance	Canada Pension Plan	Registered Pension Plan	Union Dues	Group Insurance	Hospitalization Plan	United Appeal
P. Reis	112.00				2.10	1.15	1.10	1.25	1.50
J. Brown	98.50				2.05	1.15	1.10	1.25	1.00
A. Roan	118.15				2.15	1.15	1.10	1.25	1.50
A. Blue	65.75				1.10	1.15	1.10	–	–

2. Determine the net pay for each employee given the above information (1 exemption for each).

3. Calculate the Canada Pension deductions for each employee when the gross earnings for the period are:

   (a) $87.83 (b) $22.83 (c) $96.50

4. Calculate the Canada Pension deductions, then the income tax deductions, and finally the net earnings of each employee:

	Ken Jones	*Miss O. Jordan*
Gross earnings	$116.50	$65.50
Exemptions	2	1
	Frank Scarlett	*Douglas Faire*
Gross earnings	$100.50	$25.00
Exemptions	5	3

5. Find out how the medical plan works in your province by writing your provincial government.

6. Under the compulsory medical plans many people feel medical (doctor's) expenses are *free* so they go to a doctor for the least little ailment. What is likely to happen if a large number of people adopt this habit?

6·6 Keeping Accounts

The employer must keep account of all earnings paid, of all deductions made from employee's earnings and all the payments made to various other firms on behalf of the employees. In addition the employer must, when paying in cash, make certain he has the correct *change* to put in each pay envelope.

Let us begin by looking at a *Currency Analysis* sheet prepared by the payroll department of a company that pays in cash (Table 1).

Table 1

Currency Analysis
October 15, 1971

NAME	NET EARNINGS	Bills					Coins				
		$20	$10	$5	$2	$1	50	25¢	10¢	5¢	1¢
Henry Ford	$131.43	6	1			1		1	1	1	3
Janet Black	85.75	4		1			1	1			
Joyce Meyers	143.26	7				3		1			1
Wm. Jeffrey	68.25	3		1	1	1		1			
Thomas Cromell	115.12	5	1	1					1		2
	$543.81	25	2	3	1	5	1	4	2	1	6

Table 2

Payroll Number	Name	Rate of Pay	Regular Hours	Overtime	Gross Pay	Canada Pension	Medical Insurance	Income Tax	Group Insurance	Union Dues	Unemployment Insurance Employees	Employer's Contribution U.I.C.	Other Deductions	Total Deductions	Net Earnings
11-1	Brost, M.	$2.35	40		$94.00	$1.48	$1.48	$4.00	$1.52	$1.63	$1.25	$1.25	$ –	$11.32	$82.68
11-2	Comst, J.	2.00	40	2	88.00	1.38	1.48	12.85	1.52	1.63	1.10	1.10	.25	20.21	67.79
11-3	Flare, K.	2.80	35		98.00	1.56	1.48	7.35	1.52	1.63	1.25	1.25	1.00	15.79	83.21
11-4	Klein, H.	2.60	40	4	119.60	1.95	1.48	18.15	1.52	1.63	1.40	1.40	1.50	27.63	91.97

Table 2 shows a section of a typical payroll summary made by the payroll department of a company.

Notice the summary shows the employer's contribution to Unemployment Insurance. It is equal to the employee's contribution. The amount is *not* deducted from the employee's earnings.

On the assembly line each man adds another part to the vehicle. The vehicle moves from one man to the next while each man stays in his own area. This system is highly efficient. A manufacturer can move one car every minute off the assembly line.

EXERCISE 6·6

1. Prepare a currency analysis sheet for making up pay envelopes for the following employees:

Name	*Net Earnings*
Henry Ford	$157.29
Janet Black	96.86
Joyce Meyers	163.71
Wm. Jeffrey	132.56
Thomas Cromell	99.99

2. Complete a payroll summary for:

   Jacques, A.

   40 hours at $1.96
   6 hours overtime at $1\frac{1}{2}$ x regular
   Income tax $5.76
   Canada Pension $1.48
   Medical Plan $1.56
   Group Insurance $1.61
   Union Dues $1.78
   Unemployment Insurance $1.38
   Other deductions $0.94

   Henri, K.

   38 hours at $2.04
   8 hours overtime at $1\frac{1}{2}$ x regular
   Income tax $6.90
   Canada Pension $2.36
   Medical Plan $1.94
   Group Insurance $1.73
   Union Dues $1.98
   Unemployment Insurance $1.38

Government's Role in our Daily Lives

Reproduced by permission from "How Canadians Govern Themselves", by Joseph Schull, Information Canada, Ottawa, 1971.

Which of the roles illustrated are paid for directly by the individual and which are taxed by the government?

3. Prepare a Currency Analysis sheet for the employees in question 2.

4. Prepare a payroll summary for each:

   (a) Gallope, V.; Production: 340 units

   Rate: 33¢ each on first 300 units
   38¢ each on all over 300 units
   No. of exemptions: 3
   Income Tax:
   Unemployment Insurance:
   Canada Pension Plan:
   Group Insurance: $0.56
   Union Dues: $1.56
   Medical Insurance: $1.32

   (b) Volponed, M. Sales: 2170 units

   Rate: 5¢ each on first 2000
   7¢ each on next 2000
   No. of exemptions: 5
   Income Tax:
   Unemployment Insurance:
   Canada Pension Plan:
   Group Insurance:
   Union Dues:
   Medical Insurance: $1.15
   Company Insurance: 3% of gross earnings

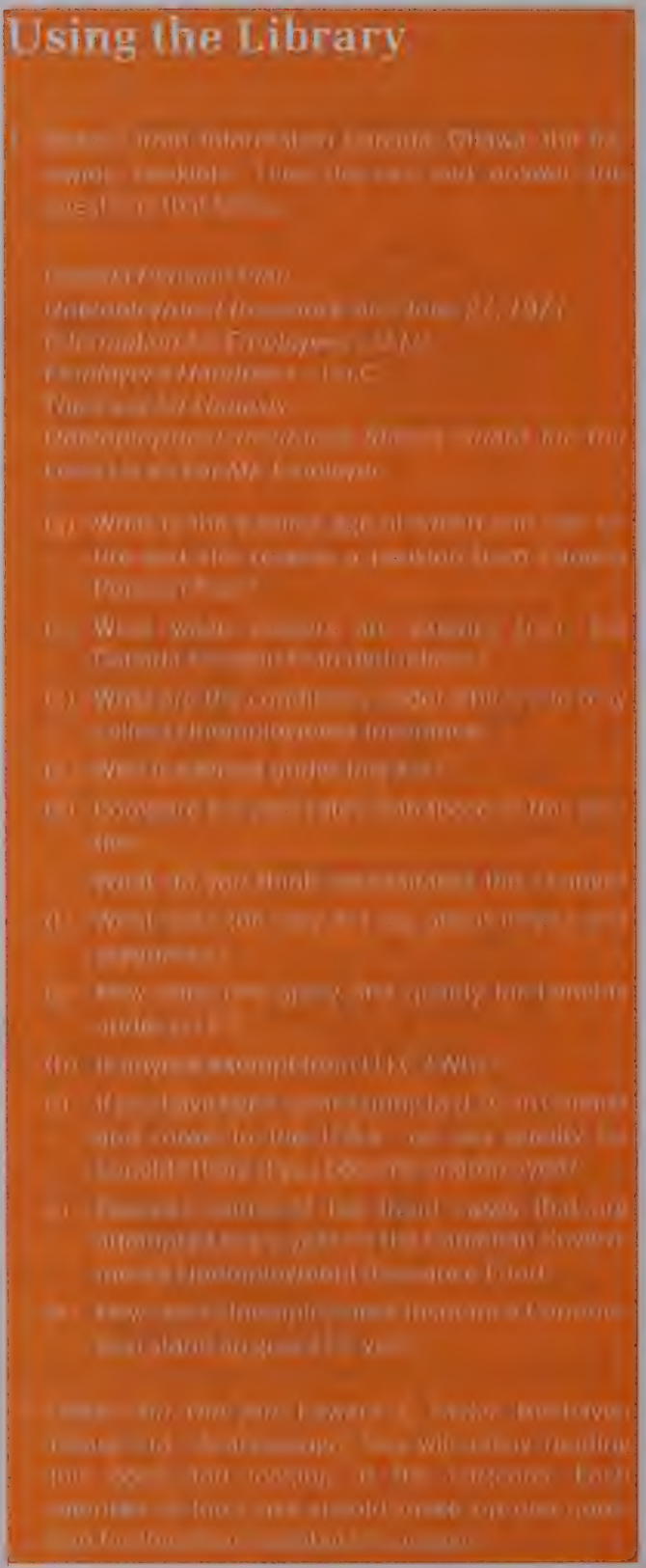
5. Arrange to visit the payroll department of a manufacturing plant or a firm that has a number of employees. Prepare a list of questions you will ask the guide showing you through the plant.

6. Arrange to visit a firm that uses a computer to handle its payroll. Notice particularly those operations still handled by employees. Do you think the computer will ever do away entirely with the human element in the payroll department? Why?

REVIEW EXERCISE 6

1. Write a review in opposition to or in defense of these three government plans: Unemployment Insurance, Canada Pension Plan, Government Health Plans. You will be assessed on the points you make that reflect your understanding of the three plans.

Using the Library

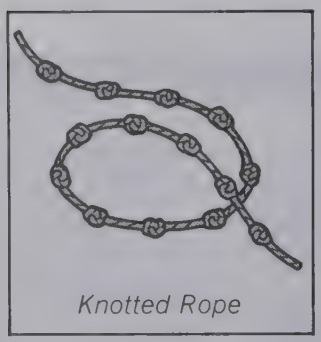


Chapter Seven
COMPUTER NUMBERS

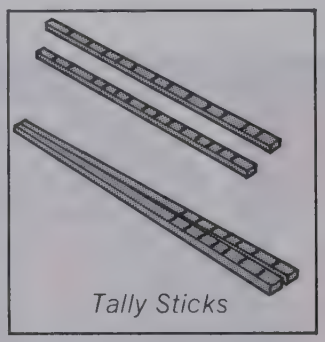
7·1 Primitive Ways of Counting

Just how our numeration system originated is not certain. Many historians have put forth theories based on known ancient systems and on systems still used today by primitive tribes.

As late as the nineteenth century tally sticks were used in the Bank of England as a method of bookkeeping.



Knotted Rope



Tally Sticks

The Incas of Peru knotted cords to record the amount of grain harvested.

Some primitive tribes of the world still do not have words in their languages for numbers greater than three. Some of these tribes count *one, two, three,* then resort to *many* for any groups greater than three.

To answer the question *How many?* it is certain man derived a great deal of help from the counters he carried with him – his fingers and toes. For example, a tribe of Indians on the Orinoco River have words for the first four numbers but use the phrases listed on this page for other numbers.

The Number	*Literal Translation*
5	a whole hand
6	one on the other hand
7	two on the other hand
10	both hands
11	one on the foot
12	two on the foot
15	a whole foot
16	one on the other foot
20	one Indian
21	one on the hand of the other Indian
40	two Indians
60	three Indians

EXERCISE 7·1

1. Write the phrase that the Indians along the Orinoco River use for
   (a) 9 (b) 14 (c) 18 (d) 30.

2. Where is the Orinoco River?

7·2 Babylonian and Egyptian Numerals

The earliest known written numerals were those used in Egypt and Mesopotamia. The people of these lands, though many miles apart, used similar methods. The Egyptian priests wrote on papyrus made from reeds while soft clay tablets were used in Mesopotamia. Let us look at the numerals used by people of various countries.

Many hundreds of clay tablets dating back to 3500 B.C. have been found that show mathematics used by the Babylonians. They were the first to divide the day into 24 hours, the hour into 60 minutes, and the minute into 60 seconds. Whereas the Egyptians used

Egyptian Numerals

1	2	3	4	5	6	7	8	9
I	II	III	IIII	III II	III III	IIII III	IIII IIII	III III III

10	∩	heelbone
100	𓍢	coiled rope in fields
1,000	𓆼	Lotus flower abundant
10,000	𓂭	painting finger
100,000	𓆑	tadpole
1,000,000	𓁨	man in astonishment

strokes to represent the first nine numerals, the Babylonians used wedge-shaped characters. The Babylonians used the same grouping arrangement as did the Egyptians.

1	2	5	9
𒐕	𒐖	𒐗	𒐙

10	11	30	60
𒌋	𒌋𒐕	𒌋𒌋𒌋	𒐕

The Babylonians used the principles of repetition, addition and a system of *place-values* based on 60. Study these examples. Notice that the symbol for 60 is the same as the symbol for 1 and that there are only two symbols used.

$38 =$ 𒌋𒌋𒌋 𒐙

$10 + 10 + 10 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$

$73 =$ 𒐕𒌋 𒐕𒐕𒐕

$60 + 10 + 1 + 1 + 1$

$162 =$ 𒐖𒐖 𒌋𒌋𒌋 𒐖𒐖

$2 \times 60 + 10 + 10 + 10 + 10 + 1 + 1$

$324{,}131 =$ 𒐕 𒌋𒌋𒌋 𒐕𒐕𒐕 𒌋𒌋𒌋 𒐕

$60^3 + 30 \times 60^2 + 3 \times 60 + 5 \times 10 + 1$

The position of the symbols indicated the value. Difficulties could be experienced. Does 𒐕 mean 1 or 60 or $60 \times 60 \times 60$? In our work here we shall assume the first symbol on the right always has its lowest place-value. Can you group differently to get a different decimal value for the above Babylonian number?

EXERCISE 7·2

1. What number does each of the following Egyptian numerals represent?

   (a) 𓆑𓆑 𓆼𓆼 𓍢𓍢 ∩∩ III III

   (b) 𓁨𓁨𓁨 𓍢𓍢 𓍢𓍢𓍢 IIII III

   (c) 𓂭𓂭𓂭 ∩∩∩

   (d) 𓆼𓆼𓆼𓆼 𓍢𓍢 ∩∩ ∩∩∩ IIII

   (e) 𓆑𓆑𓆑 𓂭𓂭 𓍢𓍢 IIII IIII

2. About 3500 B.C. an Egyptian scribe recorded that 120,000 prisoners, 400,500 oxen and 1,422,600 goats were captured in a battle. Write Egyptian numerals for the underlined numbers.

3. What number does each of the following Babylonian numerals represent?

   (a) 𒌋𒌋 𒐕𒐕𒐕

   (b) 𒌋𒌋𒌋 𒐖𒐖

   (c) 𒐕 𒌋𒌋𒌋 𒐕𒐕𒐕𒐕

   (d) 𒌋𒌋 𒐕𒐕𒐕 𒌋 𒐕𒐕𒐕𒐕

   (e) 𒌋𒌋𒐕𒐕𒐕 𒌋 𒐕𒐕𒐕𒐕

   Hint! The grouping may help you.

   Consider (d) and (e). What problems do you see in the Babylonian system?

4. Write each in Babylonian numerals.

   (a) 16 (b) 23 (c) 85

   (d) 130 (e) 4,461 (f) 246,033

   Compare answers. What problems do you see?

7·3 Maya System

The Maya system was developed by a people of Central America who were cut off from the civilizations of Europe and Asia. Notice the similarity to the systems mentioned above. The dot, stroke and oval were the symbols used. The numbers one to nineteen inclusive were built-up by using the dot and the stroke.

1	2	3	4	5	6
•	••	•••	••••	—	• —
9	10	11	15	16	19
•••• —	 =	• =	 ≡	• ≡	•••• ≡

In our system, the decimal system, adding a zero to a numeral makes the number it represents 10 times larger (40 is 10 times larger than 4). The Mayas used the oval (⬭) to make a number 20 times larger.

20	40	80	100	180	200	5040
• ⬭	•• ⬭	•••• ⬭	— ⬭	•••• — ⬭	 = ⬭	•••• = ⬭ ⬭

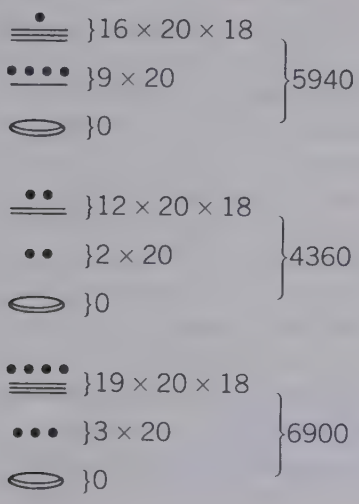
This small serpentine figurine of a kneeling man, approximately five inches high, comes from a Late Classic tomb at Altun Ha, dating from about A.D. 650. The object may be much older than the tomb in which it was found. Identification of the figure is uncertain, but it may represent a supplicant to the Maya gods.

The Temple of the Masonry Altars, Altun Ha, British Honduras (Belize). Probably dedicated to the Sun God, the 65-foot-high temple was completed in this form about A.D. 600. Ceremonial processions probably ascended the stairway to a small altar at the top, where unusual sacrifices of resin and jade objects were cast into blazing fires.

The Mayas adjusted this system to fit their moon calendar of 360 days. The second oval multiplied the number by 18. Hence in time reckoning:

was $2 \times 20 \times 18$ not $2 \times 20 \times 20$.

Study the illustrations. Notice the place-value used depended on a vertical arrangement rather than a horizontal arrangement.



EXERCISE 7·3

1. Write the decimal numeral that is equivalent to each of the following Maya numerals.

(a) (b) (c)

(d) (e) (f)

2. Write Maya numerals equivalent to these decimal numerals.

(a) 14 (b) 19 (c) 24

(d) 140 (c) 536 (f) 3600

7·4 Roman Numerals

The modern Roman Numerals have symbols for one, five, ten, fifty, one hundred, five hundred, and one thousand. There are three rules that are used with the symbols.

The symbols are:

1	5	10	50	100	500	1000
I	V	X	L	C	D	M

The special rules are:

I may be subtracted from only V and X.

X may be subtracted from only L and C.

C may be subtracted from only D and M.

When I, X or C appears to the left of a symbol of greater value, it is to be subtracted from the greater value.

Examples

IV = 4 IX = 9
XL = 40 XC = 90
CD = 400 CM = 900

Cornerstone. Central Library, Toronto.
When was it built?

EXERCISE 7·4

1. The Roman numeral for 4 is IV or 1 subtracted from 5. This illustrates the principle of subtraction. Use the principles of subtraction and addition to write the Roman numeral for each:
   (a) 43 (b) 49 (c) 94 (d) 98
   (e) 99 (f) 440 (g) 490 (h) 499
   (i) 945 (j) 993 (k) 987 (l) 999.

2. Write a decimal numeral equivalent to each:
   (a) XXVII (b) CCXLVI
   (c) DLXXVIII (d) MCMLXX.

7·5 Hindu-Arabic Numerals

The numeration system we use today is often called the Arabic system. It is a mistaken notion that the Arabs invented the system – in fact, the Arabs never have used the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Historians have pieced together the history of our numeration system. Centuries before the Roman Empire reached its peak mathematicians along the Indus River had found a close value for π. Sometime before 500 B.C., it is believed, the Hindus let the two horizontal lines = become Z and ≡ become 3. Thus the origin of 2 and 3. The nine symbols used by the early Hindus were much like these:

1 2 3 4 5 6 7 8 9

Some unknown person decided to use an abacus-like instrument in which he let the beads in the groove on the right represent units, those in the middle groove the tens and those in its left groove represent the hundreds. Hence the number shown in Figure 7-1 is

$$5 \times 100 + 2 \times 10 + 3 \times 1 = 523.$$

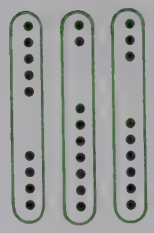


Figure 7-1

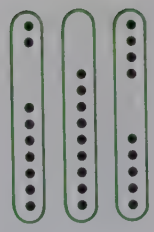


Figure 7-2

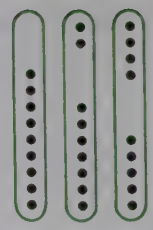


Figure 7-3

How should we write the numeral for this number in Figure 7-2? (We will use our present symbols.) Should we write 24? Surely this is not the same 24 as in Figure 7-3. Someone, possibly a clerk, decided to use a dot between the 2 and 4 in the case of Figure 7-2: 2.4. This meant 2 × 100 + no tens + 4 × 1. Later in history – sometime after the fall of the Roman Empire – the zero as we know it today came into use. Evidence now indicates the Hindu system spread rapidly by means of a text written by an Arab scholar who had learned the system in India. Merchants carried the news of the new system to Africa, the Mediterranean countries, and throughout Europe. It soon won out over the Roman numerals. Since it was spread by a translation in the Arabic language, it was known as the Arabic numeral system.

Evolution of the Hindu-Arabic Numerals

About

1000 A.D.

1200

1300

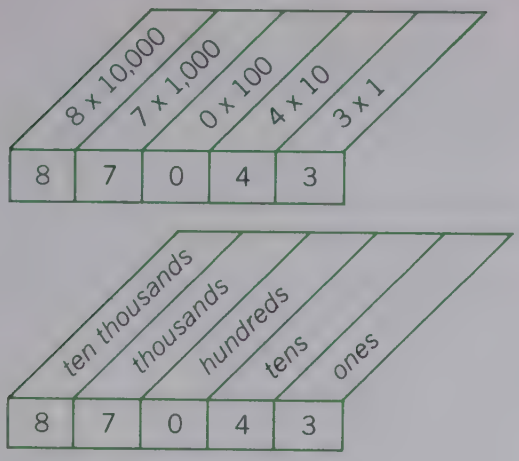
1400

1500

The Hindu-Arabic system employed ten different symbols called *digits* which meant *fingers*, and two principles:
(a) place-value principle,

(b) additive principle.

The meaning of the number 87,043 is



place-values

EXERCISE 7·5

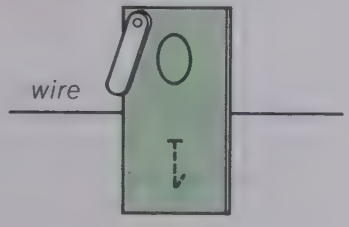
1. Throughout the study of the numeral systems we see the prominent role played by the number of fingers a man has. Devise a number system which may have evolved had man had 8 fingers.

2. What is the meaning of the decimal numeral
   (a) 34,567 (b) 504,032?

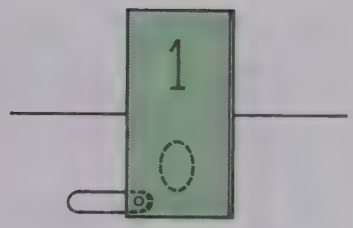
7·6 A New Notation

Let us set up a number system. Study the apparatus shown here. Each *flap* is free to turn on the wire when pressure is applied. Each turns independently except when the *trigger* that flips out is already touching another flap.

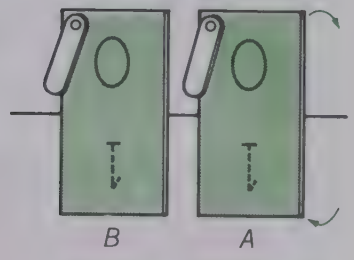
When the flap is in this position the trigger is in.



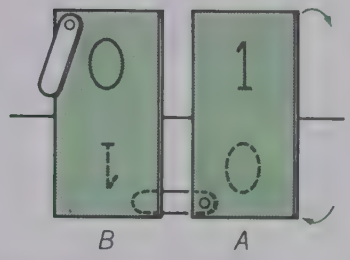
When the flap is in this position the trigger is out.



Flap *A* can flip without changing flap *B*.

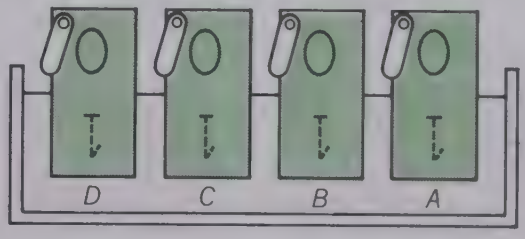


If flap *A* is flipped now it will cause flap *B* to flip.

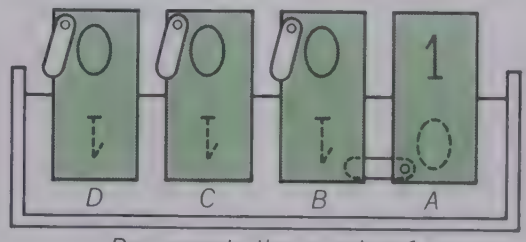


Each flap has two symbols: '0' on one side and '1' on the other side as shown above. Note that the flap, in flipping, turns such that the top goes away from us and the bottom comes towards us.

Now let us make a number of flips and count them. We start like this.

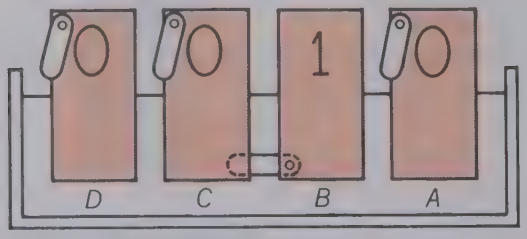


Flip flap *A*. This now shows the number 1. (one flip).



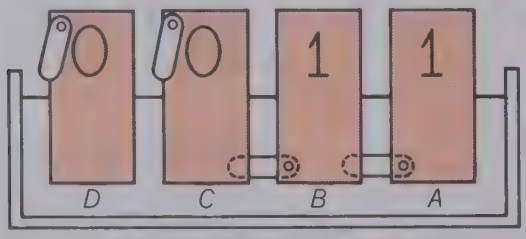
Represents the number 1.

Flip flap *A* again. This shows the number 2 (two flips).



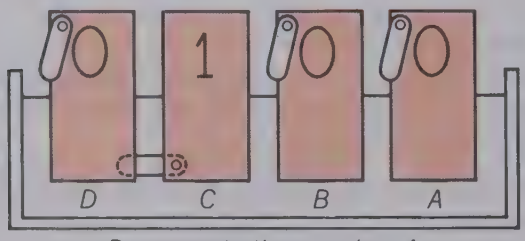
Represents the number 2.

Notice flap *A* caused flap *B* to flip.
Now flip flap *A* again.



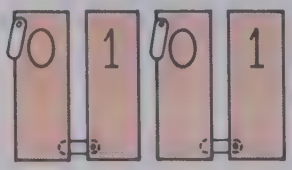
Represents the number 3.

This shows the result of 3 flips.
Now flip flap *A* again.



Represents the number 4.

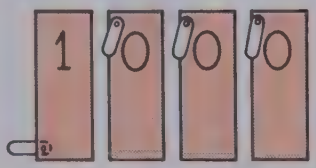
This shows the result of 4 flips.



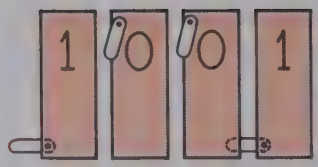
Represents the number 5.

Represents the number 6.

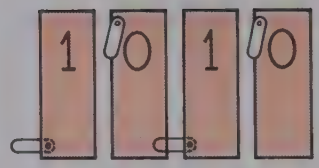
Represents the number 7.



Represents the number 8.



Represents the number 9.



Represents the number 10.

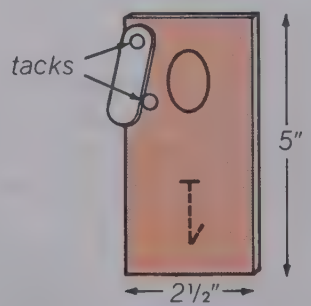
Let us summarize the symbols used for the 11 numbers.

There are only 2 digits used, 1 and 0. Therefore we call this system a *binary* system (*bi* means two),

0	0 0 0 0
1	0 0 0 1
2	0 0 1 0
3	0 0 1 1
4	0 1 0 0
5	0 1 0 1
6	0 1 1 0
7	0 1 1 1
8	1 0 0 0
9	1 0 0 1
10	1 0 1 0

EXERCISE 7·6

1. Construct a flip-flap counter with 5 flaps. Drill the hole in each flap a little smaller than the wire so that each flap can turn but does so only when force is applied. The triggers must be loosely attached so they fall out or in by their own weight.



2. Write the numerals for the numbers 1 through 20 in binary notation. Try to think out the system. Do not look at the previous material unless necessary. If necessary, draw or use the flip-flap counter. You

should learn this notation since you will be doing a great deal of work in this system.

3. Write the numbers 1, 2, 4, 8 and 16 in binary notation. Notice that each binary numeral has only one '1'. Refer to the counter above. In which position does the numeral 1 occur for the number 1? 2? 4? 8? 16?

4. What is the value of 2^0? 2^1? 2^2? 2^3? 2^4?

7·7 Binary Notation

In our numeral system in the previous section we found

$$\begin{aligned} 0\,0\,0\,0\,1 &\equiv 1 \\ 0\,0\,0\,1\,0 &\equiv 2 \\ 0\,0\,1\,0\,0 &\equiv 4 \\ 0\,1\,0\,0\,0 &\equiv 8 \\ 1\,0\,0\,0\,0 &\equiv 16 \end{aligned}$$

In this system when we write 1 1 1 1 1 we mean $16 + 8 + 4 + 2 + 1$. We say the first position on the right is the '1' position, the second from the right is the '2' position, the third from the right is the '4' position, and so on. This is shown in Figure 7-4.

Figure 7-4

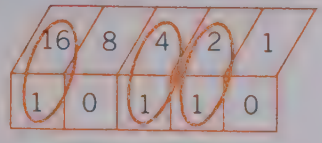


Figure 7-5

The numeral 10110 represents $16 + 4 + 2$ or 22.

Example 1

What does the numeral 11001 represent?

Solution

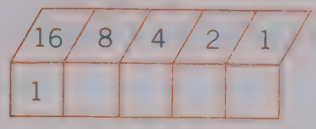
Starting from the right in 11001 we have a '1' in the first place. The next two zeros mean we do not have a '2' nor a '4'. The '1' in the fourth place means we have an '8'. The next '1' indicates we have a '16'. Therefore altogether we have $16 + 8 + 1$ or 25.

Example 2

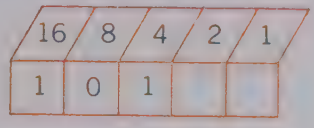
Write 20 in binary notation without referring to the tables above.

Solution

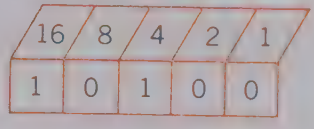
Write down the place-value chart. Under the '16' place a '1'.



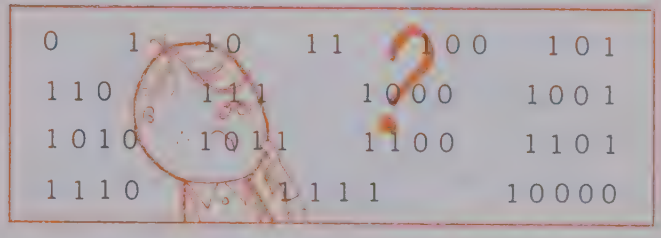
This means we have 16. We need 4 more. Therefore we place a '0' under the '8' and a '1' under the '4'.



At this point we have 16 + 4 or 20. Since we do not need any more we place a '0' in each remaining place.



Therefore 20 in binary notation is 10100.



EXERCISE 7·7

1. The numerals in this question are in binary notation.

   Write the equivalent decimal numeral for each.

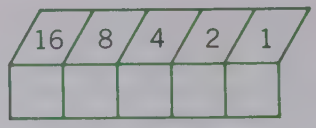
   (a) 00011 (b) 01101
   (c) 01011 (d) 10001
   (e) 11011 (f) 10111
   (g) 11000 (h) 11110

2. Write each decimal numeral in binary notation.
   (a) 15 (b) 6 (c) 12 (d) 31
   (e) 28 (f) 18 (g) 10 (h) 26

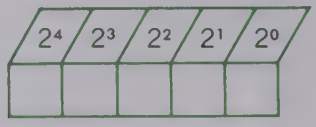
3. Write your age in binary notation.

7·8 Large Numbers

So far we have restricted our numbers in the binary notation to those natural numbers less than 32. Let us enlarge our system so that we can write large numbers. You will recall the place-values are:



Let us write each of these numbers as powers of 2. $16 = 2^4$; $8 = 2^3$; $4 = 2^2$; $2 = 2$; $1 = 2^0$. Therefore our place-value chart may be written this way:



We can use this as a pattern to extend our place-value chart.

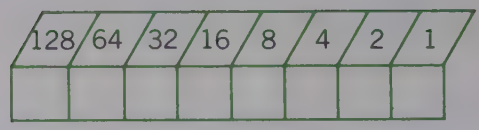
We know that $2^6 = 32$, $2^7 = 64$ and $2^8 = 128$. Study the following examples to see how we use this extended chart.

Example 1

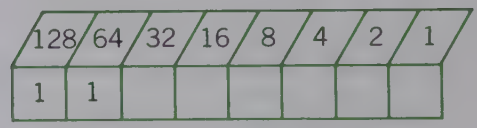
Write 205 in binary notation.

Solution

Draw the place-value chart. (Eventually you will not have to draw it because you will remember it.)

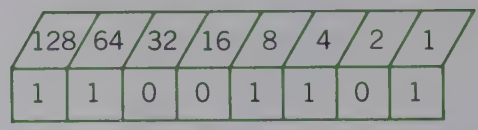


Start with 128. Then add as many of the other values as you can until you have a sum of 205.



$205 = 128 + \blacksquare$
$205 = 128 + 64 + \blacksquare$

We cannot use 32 or 16 for if we do our sum will be greater than 205. Therefore we place a '0' in each of these two places.



$205 = 128 + 64 + 8 + \blacksquare$
$205 = 128 + 64 + 8 + 4 + \blacksquare$
$205 = 128 + 64 + 8 + 4 + 1$

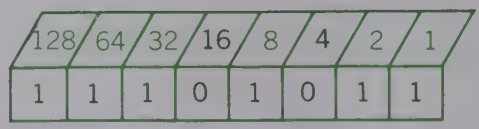
In binary notation 205 is written 11001101.

Example 2

What decimal numeral is equivalent to 11101011 in binary notation?

Solution

Draw the place-value chart and place the numerals in it.



Find the sum of the numerals in color.

$$128 + 64 + 32 + 8 + 2 + 1 = 235.$$

$\therefore$ 235 is the decimal numeral equivalent to the binary numeral 11101011.

How does a computer affect the lives of these people?

EXERCISE 7·8

1. What decimal numeral is equivalent to each binary numeral?
   (a) 11011100 (b) 10111100
   (c) 11000111 (d) 11111000
   (e) 10000000 (f) 10101010
   (g) 10001000 (h) 11111111

2. Write in binary notation:
   (a) 250 (b) 159 (c) 175
   (d) 142 (e) 151 (f) 169
   (g) 130 (h) 222 (i) 200.

7·9 Different Bases

The numeration system you are most familiar with is called the decimal system or base 10 system. The decimal system has 10 different digits, hence the term *base 10*. In base 10, you will recall, the numeral 4562 may be expanded to

$$4 \times 10^3 + 5 \times 10^2 + 6 \times 10^1 + 2 \times 10^0.$$

The numeral 50,403 may be expanded to

$$5 \times 10^4 + 0 \times 10^3 + 4 \times 10^2 + 0 \times 10^1 + 3 \times 10^0.$$

The binary system has only 2 different digits and is often referred to as the base 2 system.

An octal system (from the root octo) has 8 different digits (0, 1, 2, 3, 4, 5, 6, 7) and is called the base 8 system. The numeral 345 in base 8 is written $345_{(8)}$ and is read "three, four, five base 8".

Example 1

Write $345_{(8)}$ in decimal notation.

Solution

We will use the same method used in base 10 and base 2 work.

Part of the place-value chart is

8^2	8^1	8^0
3	4	5

or

64	8	1
3	4	5

Thus we have $3 \times 64 + 4 \times 8 + 5 \times 1 = 229$.

$$\therefore\ 345_{(8)} = 229_{(10)}$$

Example 2

Write $456_{(10)}$ in base 8 numeration.

Solution

8^3	8^2	8^1	8^0
	7	1	0

$$64 \overline{)456} \quad 7, \quad \frac{448}{8}$$

$$8 \overline{)8} \quad 1, \quad \frac{8}{0}$$

$$7 \times 64 + 1 \times 8 + 0 \times 1$$
$$= 448 + 8 + 0 = 456.$$

Example 3

Write $512_{(10)}$ in base 6 numeration.

Solution

6^3	6^2	6^1	6^0
(216)	(36)	(6)	(1)
2	2	1	2

$$216\overline{)512}\ 2 \qquad \frac{432}{80}$$

$$36\overline{)80}\ 2 \qquad \frac{72}{8}$$

$$6\overline{)8}\ 1 \qquad \frac{6}{2}$$

$$1\overline{)2}\ 2 \qquad \frac{2}{0}$$

$$2 \times 216 + 2 \times 36 + 1 \times 6 + 2 \times 1$$
$$= 432 + 72 + 6 + 2 = 512$$

EXERCISE 7·9

1. Write each in base 10 numeration.
   (a) $231_{(8)}$ (b) $336_{(8)}$ (c) $117_{(8)}$
   (d) $651_{(7)}$ (e) $2113_{(4)}$ (f) $423_{(5)}$
   (g) $1115_{(6)}$ (h) $231_{(9)}$ (i) $5118_{(9)}$

2. Write each in base 8 notation and check.
   (a) $123_{(10)}$ (b) $3214_{(10)}$ (c) $4147_{(10)}$

3. Write each in base 6 notation and check.
   (a) $54_{(10)}$ (b) $43_{(10)}$ (c) $794_{(10)}$

4. Copy and complete the table.

base 10	base 2	base 4	base 8
1			
2			
3			
⋮			
32			

5. Write each first in base 10 then in base 8.

base 2	base 10	base 8
101		
111		
1001		
1101		
11110		
11111		
111101		

7·10 Computers

The most practical use of the binary notation is in the computer. A computer is designed such that when a switch is *on* (a light is on) it represents the digit 1 and when a switch is *off* (no light) it represents the digit 0 (zero). Study the electric circuit to see how we can represent binary numbers.

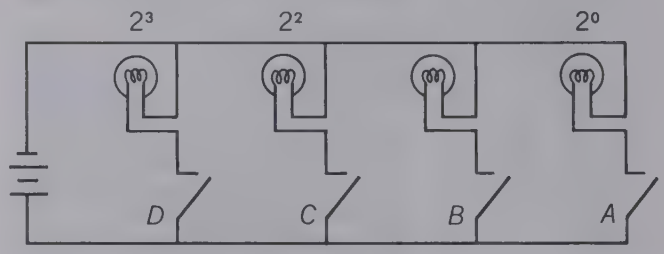


Figure 7-6

We let each light represent a place-value as shown in Figure 7-6. All switches are open. Therefore the numeral is

$0000_{(2)}$.

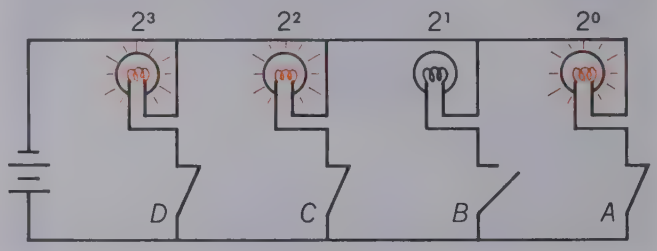
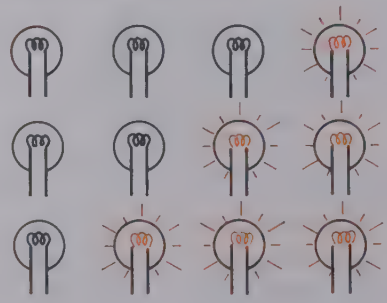


Figure 7-7

Figure 7-7 shows switches *D*, *C* and *A* closed. The lights show the numeral $1101_{(2)}$, or $13_{(10)}$ since
$(1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0)_{(10)} = 13_{(10)}$.

Shown below are other light arrangements with their numeral equivalents.



$0001_{(2)}$ $1_{(10)}$

$0011_{(2)}$ $3_{(10)}$

$0111_{(2)}$ $7_{(10)}$

EXERCISE 7·10

1. Construct the circuit shown in Figure 7-6. Practise with it. Then do the following question.

2. The circuit in Figure 7-6 can be used to show the binary numerals equivalent to the base 10 numerals from 0 to 15. Draw diagrams to show the light arrangements for these numerals.

7·11 Binary Coded Decimal

Since computers depend on *on-off* switches, it is necessary either to convert all base 10 numerals to base 2 numerals or to use a code which in effect expresses base 10 numerals in terms of binary numerals. It is sometimes advantageous to use such a code. One of these codes is the *Binary Coded Decimal Code* called the B.C.D. code. The alphabet can be expressed in terms of binary numerals by this code also.

The idea behind the B.C.D. code is that each digit in a base 10 numeral is expressed by a four-place binary numeral. For example, $7_{(10)}$ represented by 0111 and $98_{(10)}$ by 1001-1000. The first four-place numeral (1001) represents the base 10 digit 9, while the second four-place numeral (1000) represents the base 10 digit 8. A hyphen was placed between the two four-place numerals only to draw your attention to the two parts. The hyphen is not used in coding. You can see that any base 10 numeral can be expressed by a combination of the four-place binary equivalents of the decimal digits 0 to 9 inclusive.

Example 1

Express the base 10 numeral 864 in B.C.D. code.

Solution

$8_{(10)}$ is represented by 1000.
$6_{(10)}$ is represented by 0110.
$4_{(10)}$ is represented by 0100.
$\therefore 864_{(10)}$ is represented by 1000 0110 0100.
Notice the spacing of the four-place numerals.

Example 2

Determine the base 10 numeral represented in B.C.D. code by 1001 0010 0011 0101.

Solution

A decoding method is shown.

1001	0010	0011	0101
9	2	3	5

Write down the coded numeral with the equivalent base 10 digit under each four-place binary numeral.
$\therefore$ the base 10 numeral is 9235.

In each four-place binary numeral, each digit is a binary digit (called a *bit*). Since there are 4 digits we are using a *4-bit* code.

Unless indicated otherwise, all numerals are base 10 numerals.

Example 3

Write 49135 in B.C.D. code.

Solution

A coding method is shown.

4	9	1	3	5
0100	1001	0001	0011	0101

Write the numeral well spaced with the equivalent four-place binary numeral under each base 10 digit.
$\therefore$ 49135 is 0100 1001 0001 0011 0101 in B.C.D. code.

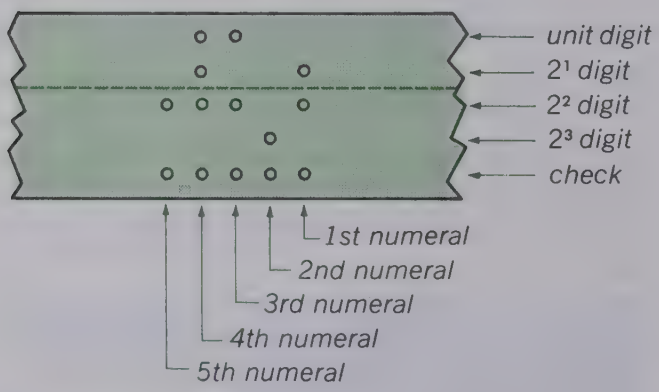
EXERCISE 7·11

1. Convert each decimal numeral to B.C.D. code.
   (a) 491 (b) 2654 (c) 37810
   (d) 746 (e) 2413 (f) 4567432

2. Convert each of the B.C.D. numerals to its equivalent decimal numeral.
   (a) 1000 0000 0100
   (b) 1001 0001 0110 0111
   (c) 1001 1000 0010
   (d) 0001 0000 0000 0000
   (e) 1001 1001 1001
   (f) 0100 1001 0010 0011 0101

3. Some computers use tape with holes punched in it. The small holes in a continuous line down the tape position the tape in the computer. The top 4 lines of larger holes represent 4-bit B.C.D. numerals. The bottom line of holes is used for checking and is to be used for spacing in this work.

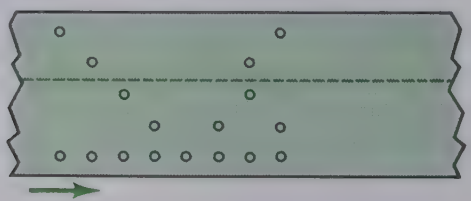
   The tape shown moves to the right. The numeral represented is 68574.



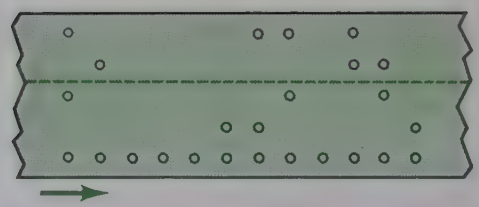
   Draw tapes that show each numeral.
   (a) 4591 (b) 6754
   (c) 8329145 (d) 4050

   Write the numeral expressed by each tape.

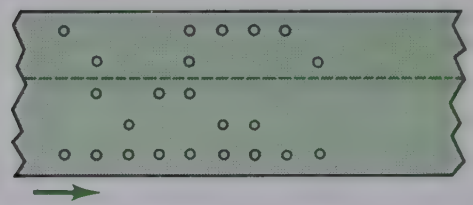
   (e)



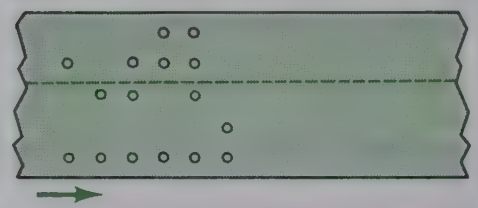
   (f)



   (g)



   (h)



7·12 Alphameric Code

We used 4-bit numerals above in the B.C.D. code. This allows 16 (2^4) different symbols to be used: the ten digits from zero to nine inclusive, and six operation signs such as +, − and =. If 6-bits are used 2^6 or 64 different symbols may be represented. Hence the 26 letters of the alphabet, the 10 digits and 28 other symbols can be represented. This produces a fairly complicated table which we will not study here but which is used by some computer programs.

A simplified version of this 6-bit code will be explained here. Each letter of the alphabet will be represented by a binary numeral. For example, in 4-bit code we may have this table:

Let A be represented by 40 which is 0100 0000
Let B be represented by 41 which is 0100 0001
⋮ ⋮ ⋮
Let J be represented by 49 which is 0100 1001
Let K be represented by 50 which is 0101 0000
⋮ ⋮ ⋮
Let Z be represented by 65 which is 0110 0101

From the context of the material it is obvious to the reader whether the B.C.D. code represents a numeral or a letter. A code in which both numerals and letters are symbolized is called an *alphameric code.*

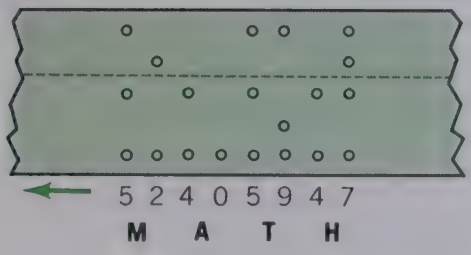
EXERCISE 7·12

1. Complete the alphameric code which was started in the section above.

2. Write the letter represented by each alphameric code when A is represented by 0100 0000.
   (a) 0101 0000 (b) 0101 1000
   (c) 0100 0111 (d) 0110 0000
   (e) 0110 0010 (f) 0100 0101

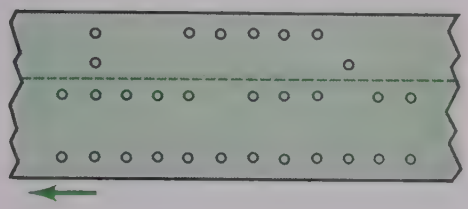
3. Write a message in alphameric code.
   Exchange messages with a person sitting near you. Decode each other's message. Write a reply in alphameric code.

4. A message can be sent on computer tape in 4-bit alphameric code. The word MATH is shown on the tape. The alphameric code starts with
   A ≡ 0100 0000.



   Write a brief message on computer tape using the alphameric code in question 1. Exchange tapes with a person near you. Write a reply in alphameric code.

5. Read the message shown here, A ≡ 0100 0000.



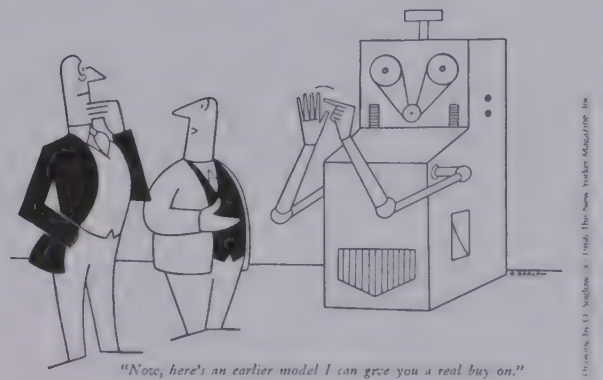
7·13 Special Challenges

1. A clock can chime at the most 4 times at each hour. It can chime either a *ding* or a *dong*.
   (a) Invent a number system using a *ding* or a *dong* so that it can chime out the hours.
   (b) If the clock can chime only 4 times with a *ding*, a *dong*, or a *dang*, invent the simplest way it can chime out the hours.

2. Read about *Napier's Bones* invented by John Napier in 1617. Make a set of Napier's bones (rods) on cardboard. Use the set to multiply 365 by 18.

3. Learn how to multiply and divide on a nomograph.

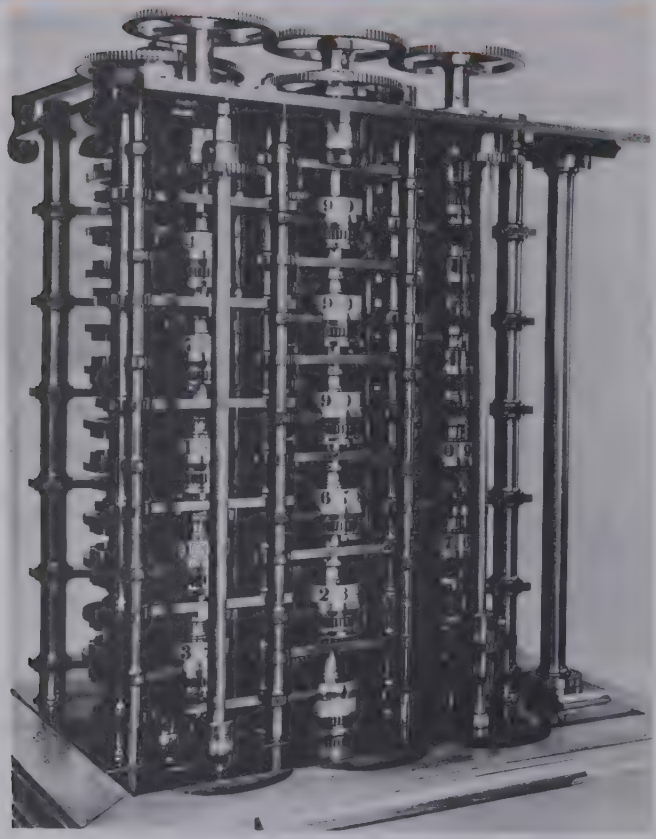
4. Investigate the various uses of computers – from computing to steering spaceships.



"Now, here's an earlier model I can give you a real buy on."

Using the Library

A modern electronic calculator.



Part of differential calculating machine on which Charles Babbage, English mathematician worked for 37 years, to find it rejected for patents by the government.

REVIEW EXERCISE 7

1. Write each as a base 10 numeral.
   (a) $1010110_{(2)}$ (b) $10010011_{(2)}$
   (c) $123_{(8)}$ (d) $352_{(6)}$

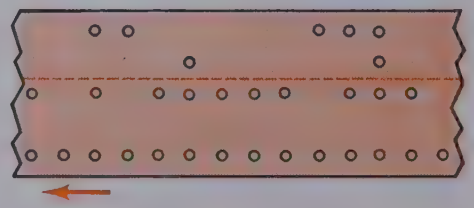
2. Write a numeral for $436_{(10)}$ in
   (a) base 2 (b) base 5 (c) base 8.

3. Write the next higher numeral in the same base.
   (a) $110111_{(2)}$ (b) $1244_{(5)}$ (c) $17777_{(8)}$

4. Write in B.C.D. code each decimal numeral given.
   (a) 23 (b) 89 (c) 2476

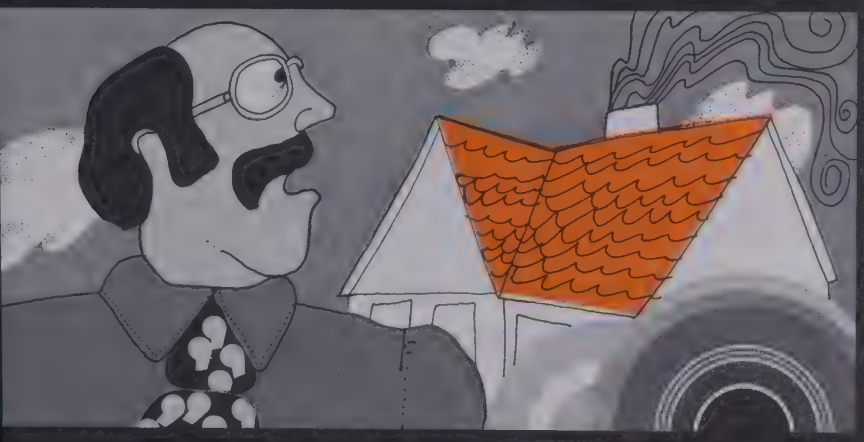
5. Write as a decimal numeral each numeral given here in B.C.D. code.
   (a) 100 000 (b) 110 000 001
   (c) 001 111 101

6. Decode this message using A ≡ 0100 0000.



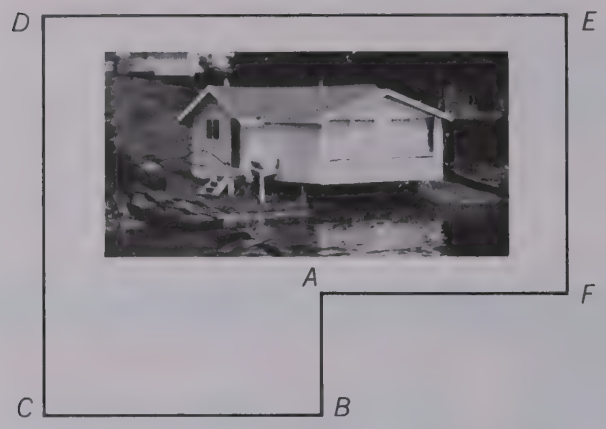
7. Write ESKIMO in alphameric code. Use A ≡ 0100 0000.

Chapter Eight
POLYNOMIALS



8·1 Using Polynomials

Mr. Bonhart and his son Keith bought a lakefront lot on which to build a cabin. It was in this shape.



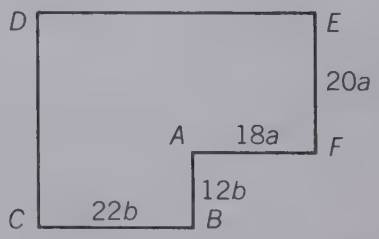
They decided to find the dimensions by pacing off the sides. Mr. Bonhart paced off A to B to C while Keith paced off A to F to E. A to B was 12 paces and B to C was 22 paces. A to F was 18 paces and F to E was 20 paces.

They knew that all the corners were right angles. They had to wait until they reached home to measure their paces.

That afternoon on the way home Keith did some preliminary calculations. He called the length of his pace a feet and the length of his father's pace b feet. He then labelled his diagram like this:

A to B is $12b$ feet
B to C is $22b$ feet
A to F is $18a$ feet
D to E is $20a$ feet

Keith wondered what the distances E to D and F to C were. Can you help Keith?



What is the perimeter of the lot?

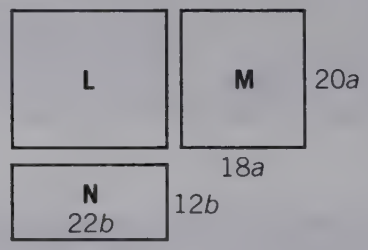
Keith wrote:

$$\text{Perimeter} = 22b + 12b + 18a + 20a + 18a + 22b + 20a + 12b.$$

Do you agree?

$$\text{Perimeter} = (68b + 76a) \text{ feet.}$$

Do you agree?

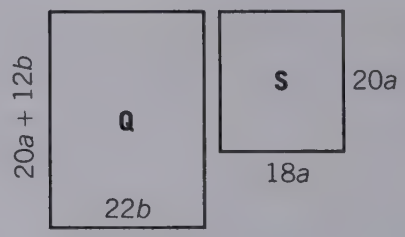


To find the area Keith wrote:

Area of region **M** $= (20a \times 18a)$
$= (360a^2)$ sq. ft.
Area of region **N** $= (22b \times 12b)$
$= (264b^2)$ sq. ft.
Area of region **L** $= (22b \times 20a)$ Why?
$= (440ba)$ sq. ft.

(1) Total area $= (360a^2 + 264b^2 + 440ba)$ sq. ft.

To check his work Keith found the area by using this diagram.



Total area = area of region **Q**
+ area of region **S**
$= 22b(20a + 12b) + 18a(20a)$

$= 22b(20a) + 22b(12b) + 18a(20a)$

The distributive principle is used for that portion in color. We distributed the first term $(22b)$ over both terms in the binomial $(20a + 12b)$.

(2) Total area $= (440ba + 264b^2 + 360a^2)$ sq. ft.

Is this the same answer as he had in statement (1)? Explain.

Keith then pondered several mathematical problems. He thought, "I have been adding polynomials. Can I subtract them?."

He wrote:

$23a + 14b - 12a - 8b = 23a - 12a + 14b - 8b$

What did he do here?

(3) $23a + 14b - 12a - 8b = 11a + 6b$

He then thought, *I can check by letting $a = 3$ and $b = 4$ and by:*

$$\begin{aligned} 23a + 14b - 12a - 8b &= 23(3) + 14(4) \\ &\quad - 12(3) - 8(4) \\ &= 69 + 56 - 36 - 32 \\ &= 57 \end{aligned}$$

Right-hand portion:

$$\begin{aligned} 11a + 6b &= 11(3) + 6(4) \\ &= 33 + 24 \\ &= 57 \end{aligned}$$

Since the left-hand side is the same as the right-hand side, statement (3) is correct.

Keith had difficulty with this problem.

He had written:

$3a + 4b - (2a - 3b) = 3a + 4b - 2a - 3b$

When he attempted to check this by using $a = 3$ and $b = 4$, it did not check. Do you see his mistake?

$$\begin{aligned} 3a + 4b - (2a + 3b) &= 3a + 4b - 1(2a - 3b) \\ &= 3a + 4b - 1(2a) - 1(-3b) \\ &= 3a + 4b - 2a + 3b \\ &= a + 7b \end{aligned}$$

Does this check now?

I have multiplied a polynomial by a monomial. Can I divide a polynomial by a monomial?

Keith wrote a problem: $\dfrac{32a - 14b}{2}$.

His solution: Note that he divided each term in the dividend by divisor.

$$\begin{aligned} \frac{32a - 14b}{2} &= \frac{32a}{2} - \frac{14b}{2} \\ &= 16a - 7b \end{aligned}$$

Check Keith's work by making these replacements:

$a = 3, \quad b = 4.$

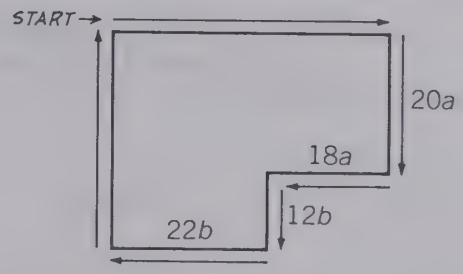
Study this division:

$$\begin{aligned} \frac{15x^2 - 4x + 26x^3}{6x} &= \frac{15x^2}{6x} - \frac{4x}{6x} + \frac{26x^3}{6x} \\ &= \tfrac{5}{2}x - \tfrac{2}{3} + \tfrac{13}{3}x^2 \end{aligned}$$

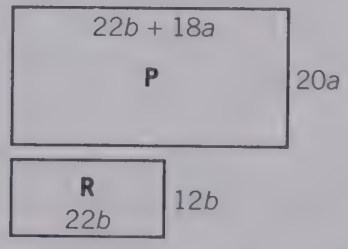
Check this work by using $x = 3$.

EXERCISE 8·1

1. Check the perimeter of the lot by adding the dimensions in the direction indicated in the diagram. Do you get the same perimeter as Keith did?



2. Check the area of the lot by finding the area when the lot is divided as in this diagram.



Total area = area of **P** + area of **R**.

3. Find the sum in each case:
   (a) $3a + 2b + 15a + 12a + 3b + 6b + 2a$
   (b) $14b + 23b + 15b + 30a + 28a + 14a + 28b$
   (c) $17x + 15y + 12y + 13x + 15y$
   (d) $29w + 12v + 17w + 17v + 9v$
   (e) $105t + 63s + s + 2t + 91s + 47t$.

4. Multiply each:
   (a) $3a(6a)$ (b) $8a(26a)$
   (c) $13b(12b)$ (d) $19x(37x)$
   (e) $4a(9a + 7b)$ (f) $5m(6n + 2m)$
   (g) $17w(15v + 69w)$ (h) $39z(56v + 42w)$.

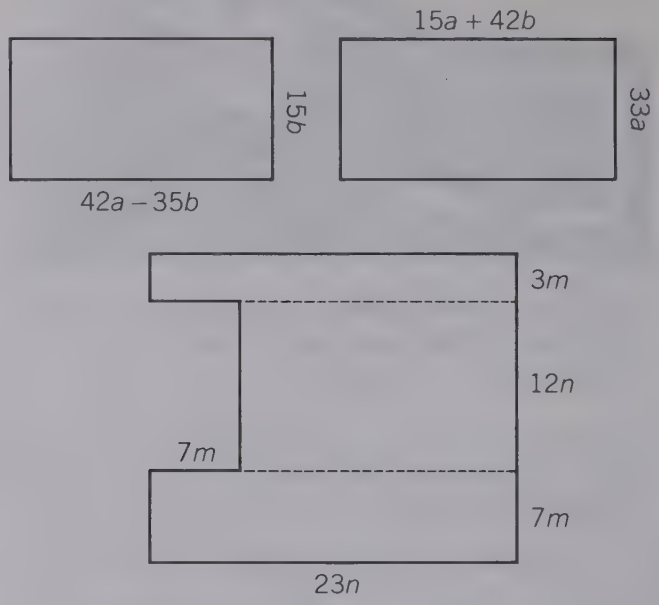
5. Find the sum and check each by using $a = 4$, $b = 5$, $c = 3$:
   (a) $12a - 3b + 5a + 7b$
   (b) $13b - 25c + (4a - 7c) - (25b - 7a)$
   (c) $14a - 13c - 6b - 7b - (3c + 15a)$
   (d) $17a - 29a + 39a + 48b - 57b + 11b$
   (e) $(4a - 5b) + 6a - (7b - 6a)$
   (f) $7a - (6a - 3b) - 7c$
   (g) $6a + 4b - (6c - 7b) + (7a + 4b)$
   (h) $7a - (3a + b) - (2c - 3b)$.

6. If $a = 2$ feet and $b = 3$ feet,
   (a) find the actual dimensions of Mr. Bonhart's lot,
   (b) calculate the actual area of the lot. How many ways can you do this? How many ways can you check this answer?

7. Divide each:
   (a) $\frac{6a - 8b}{2}$ (b) $\frac{42x - 21y}{7}$
   (c) $\frac{144w - 108v}{12}$ (d) $\frac{4a^2 - 8a}{2a}$
   (e) $\frac{25a^2 - 15a}{5a}$ (f) $\frac{33a^2}{22a}$
   (g) $\frac{14m - 21m^2}{3m}$ (h) $\frac{18y + 26y^2}{36y}$
   (i) $\frac{44a^3 - 32a^2 + 16a}{16a}$ (j) $\frac{144w^2 - 108w - 72}{36w}$

8. Simplify:
   (a) $\frac{32a - 16a^2}{8a} + \frac{24a - 9}{3} - (6a + 1)$
   (b) $\frac{15a - 21a^2}{3a} - \left(\frac{14a^2 - 21a}{7a}\right) - (7a - 2)$.

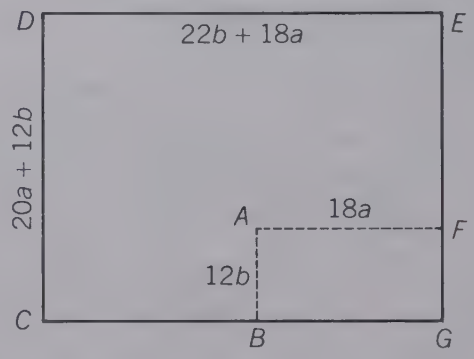
9. Find the perimeter and area of each:



8·2 Short Cut to Multiplying

Keith might have found the area of his lot by another method.

Area of lot = area of rectangle *DEGC*
– area of rectangle *AFGB*



(1) Area of lot = $(22b + 18a)(20a + 12b)$
$- (18a)(12b)$

Keith would need to know how to multiply this:
$(22b + 18a)(20a + 12b)$.

Multiply both *terms* in the *second binomial* by the *first term* in the *first binomial*.

$(22b + 18a)(20a + 12b) = 22b(20a) + 22b(12b) + \cdots$

Then multiply both terms in the second binomial by the *second term* in the first binomial:

$(22b + 18a)(20a + 12b)$

$= 22b(20a) + 22b(12b) + 18a(20a) + 18a(12b)$
$= 440ba + 264b^2 + 360a^2 + 216ab$
$= 264b^2 + 360a^2 + 656ab$

since ab is the same as ba.

Then Keith put this into sentence (1).

(2) Area of lot $= 264b^2 + 360a^2 + 656ab - 216ab$
$= 264b^2 + 360a^2 + 440ab$

Is this the same as Keith got in Section 8·1? Check.

Example 1

Multiply and check: $(3a - 2b)(4a + 5b)$

Solution

(1) $(3a - 2b)(4a + 5b) = 3a(4a) + 3a(5b) + \cdots$

(2) $(3a - 2b)(4a + 5b) = 12a^2 + 15ab + (-2b)(4a) + (-2b)(5b)$
$= 12a^2 + 15ab - 8ba - 10b^2$
$= 12a^2 + 7ab - 10b^2$

In steps (1) and (2) we used the distributive principle: we distributed each term in the first binomial over each term in the second binomial.

Check by using $a = 3$, $b = 4$

Left-hand side:

$(3a - 2b)(4a + 5b) = [3(3) - 2(4)][4(3) + 5(4)]$
$= (9 - 8)(12 + 20)$
$= (1)(32)$
$= 32$

Right-hand side:

$12a^2 + 7ab - 10b^2 = 12(3)^2 + 7(3)(4) - 10(4)^2$
$= 12(9) + 7(12) - 10(16)$
$= 108 + 84 - 160$
$= 32$

The left-hand side has the same value as the right-hand side, therefore our work must be correct.

Do questions 1 through 8 in Exercise 8·2.

The FOIL Rule

Note the steps in this multiplication:

$(x + 2y)(x + 3y) = (x)(x) + \cdots$ First terms

Outside terms

$(x + 2y)(x + 3y) = x^2 + x(3y) + (2y)(x) + \cdots$

Inside terms — Sums of products of Outside terms and Inside terms

$(x + 2y)(x + 3y) = x^2 + 5xy + (2y)(3y)$

Last terms — Product of Last terms

$= x^2 + 5xy + 6y^2$

You will note that this is a simplified version of the distributive principle.

Example 2

Use the FOIL RULE to multiply each:

(a) $(x + 5y)(x + 6y)$
(b) $(x - 2b)(x + 3b)$
(c) $(2x - 2y)(3x - 5y)$.

Solution

(a) $(x + 5y)(x + 6y) = x^2 + \ldots$ First terms
$= x^2 + (6xy + 5xy) + \ldots$
Outside and Inside terms
$= x^2 + 11xy + \ldots$
$= x^2 + 11xy + 30y^2$ } Last terms
F + (O + I) + L

(b) $(x - 2b)(x + 3b) = x^2 + \ldots$ First terms
$= x^2 + 3xb + (-2b)(x) + \ldots$
Outside and Inside terms
$= x^2 + 3xb - 2xb + \ldots$
$= x^2 + xb + (-2b)(3b)$
F + (O + I) + L
$= x^2 + xb - 6b^2$

(c) $(2x - 2y)(3x - 5y)$
$= (2x)(3x) + (2x)(-5y) + (-2y)(3x) + (-2y)(-5y)$
F + O + I + L
$= 6x^2 + (-10xy) + (-6xy) + (10y^2)$
$= 6x^2 - 16xy + 10y^2$

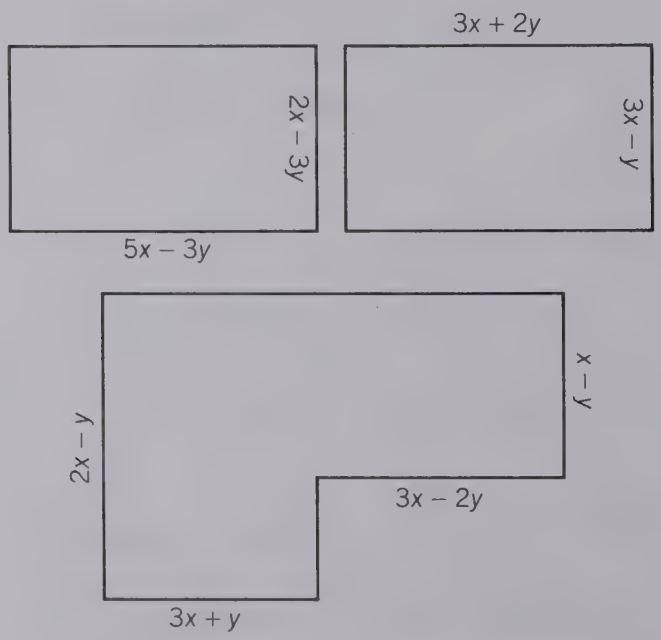
EXERCISE 8·2

Multiply and check each:

1. $(a + 3b)(a + 5b)$
2. $(a + 5b)(a + 6b)$
3. $(2a - 3b)(a + 6b)$
4. $(3a - 7b)(4a + 9b)$
5. $(8x + 6y)(3x - 5y)$
6. $(7x + 2y)(6x - 3y)$
7. $(3m - 15n)(4m - 2n)$
8. $(6m - 8n)(7m - 3n)$

Use the FOIL RULE to multiply each:

9. $(a + 2b)(a + 5b)$
10. $(a + 7c)(a + 6c)$
11. $(x - 2y)(x + 5y)$
12. $(x - 5y)(x + 7y)$
13. $(2m + 3n)(4m - 5n)$
14. $(12p + 6r)(4p - 8r)$
15. $(8p - 6r)(9p - 3r)$
16. $(8p - 3r)(5p - 7r)$
17. $\left(\frac{p}{4} - \frac{r}{2}\right)\left(\frac{p}{2} + \frac{r}{4}\right)$
18. $\left(\frac{3}{4}p + \frac{1}{2}r\right)\left(\frac{1}{2}p - \frac{3}{4}r\right)$
19. $\left(\frac{1}{2}x - \frac{1}{3}y\right)\left(\frac{3}{2}x - \frac{2}{3}y\right)$
20. $\left(\frac{4}{3}x - \frac{3}{2}y\right)\left(\frac{3}{2}x + \frac{4}{3}y\right)$
21. Find the area of each:



8·3 Factoring Polynomials

Study Figures 8-1 and 8-2 below. What is the length of each? Do you agree side DC is 20 feet and side GH is m feet since

$$160 = 8 \times 20$$

and

$$mn = n \times m.$$

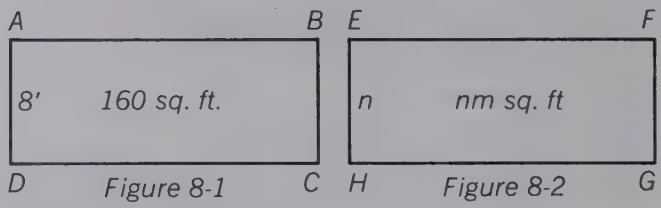


Figure 8-1 *Figure 8-2*

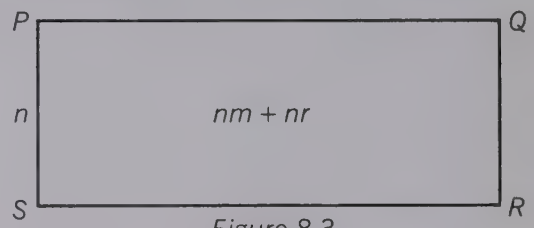


Figure 8-3

Now look at Figure 8-3. What is the length of side PQ? Do you agree it is $(m + r)$?

$$mn + nr = n(m + r)$$

In Figure 8-1 we factored 160 into two factors 8 and 20.

In Figure 8-2 we factored nm into two factors n and m.

In Figure 8-3 we factored $nm + nr$ into two factors n and $(m + r)$.

Recall how you used the distributive principle to find the product in each of these cases:

(i) $a(x + y) = ax + ay$

(ii) $p(a + b + c) = pa + pb + pc$

(iii) $m(n - p - q) = mn - mp - mq$

To find the highest common monomial factor of a polynomial we may reverse this process and use the distributive principle in reverse.

(i) $ax + ay = a(x + y)$

(ii) $pa + pb + pc = p(a + b + c)$

(iii) $mn - mp - mq = m(n - p - q)$

Do questions 1 and 2 in Exercise 8·3.

Note the following examples in which we first determine the highest common factor of the numerical coefficients then the highest common factor (HCF) of the literal coefficients. First we will look at multiplication:

(iv) $3a(2x - 5y) = 6ax - 15ay$
(v) $4a^2x(3ax - 2a^2y + 5z)$
$= 12a^3x^2 - 8a^4xy + 20a^2xz$

Now look at factoring:

(iv) $6ax - 15ay = 3a(2x - 5y)$
(v) $12a^3x^2 - 8a^4xy + 20a^2xz$
$= 4a^2x(3ax - 2a^2y + 5z)$

Note in (iv) 3 is the HCF of 6 and 15 and a is the HCF of ax and ay. Therefore $3a$ is the HCF of $6ax - 15ay$. Similarly in (v) 4 is the HCF of 12, 8, and 20 while a^2x is the HCF of a^3x^2, a^4xy, and a^2xz. Therefore $4a^2x$ is the HCF of that polynomial. In each case we get the other factor by dividing the polynomial by the respective HCF.

Example 1

Factor each:
(a) $6x^2b - 9xc - 15xd$
(b) $2a(x + y) - 3b(x + y)$

Solution

(a) The HCF of 6, 9, and 15 is 3.
The HCF of x^2b, xc, and xd is x.
Therefore the HCF of the polynomial is $3x$.
To obtain the other factor divide the polynomial by $3x$.

$$\frac{6x^2b - 9xc - 15xd}{3x} = \frac{6x^2b}{3x} - \frac{9xc}{3x} - \frac{15xd}{3x}$$
$$= 2xb - 3c - 5d$$

Hence, $6x^2b - 9xc - 15xd = 3x(2xd - 3c - 5d)$

(b) There is no common factor of 2 and 3.
The HCF of $a(x + y)$ and $b(x + y)$ is $(x + y)$.
The HCF of the polynomial is $(x + y)$.
Divide:

$$\frac{2a(x + y) - 3b(x + y)}{(x + y)} = \frac{2a(x + y)}{(x + y)} - \frac{3b(x + y)}{(x + y)}$$

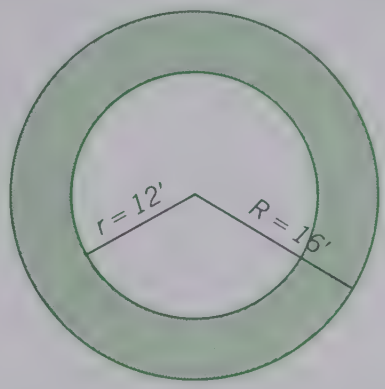
Hence
$$2a(x + y) - 3b(x + y) = (x + y)(2a - 3b)$$

Do question 3 in Exercise 8·3.

We can use this type of factoring to assist us in calculations. Study the next example.

Example 2

A gardener wishes to place a concrete walk around his flower bed as shown. Calculate the area of the path.

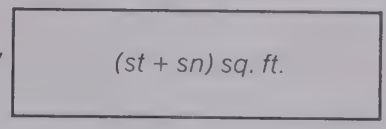


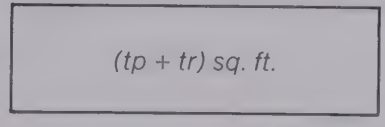
Solution

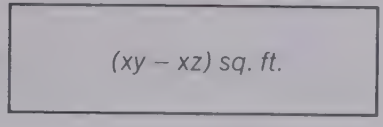
Area of ring = area of large circle
– area of small circle
$= \pi R^2 - \pi r^2$
$= \pi(R^2 - r^2)$
$= 3.14(16^2 - 12^2)$
$= 3.14(256 - 144)$
$= 3.14(112)$
$= 352$ (slide rule)
The area is 352 square feet.

EXERCISE 8·3

1. Find the length of each rectangle.

(a)


(b)


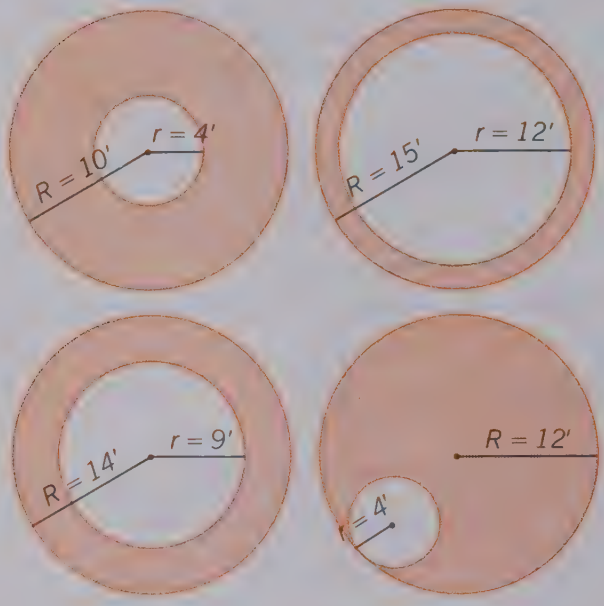
(c)


2. Factor each:
   (a) $xa - xb$ (b) $ms - mt - mr$
   (c) $tr - ts + tp$ (d) $xt - yt$
   (e) $zt - xt + tv$ (f) $mn - tn + ns$
   (g) $\frac{1}{2}hr - \frac{1}{2}ht$ (h) $\frac{2}{3}R - \frac{2}{3}R^2$

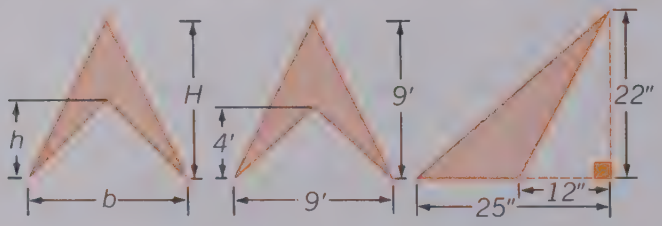
3. Find the highest common monomial factor and write two factors in each of the following:
   (a) $8 + 12a + 16a^2$ (b) $6ab^2c - 12abc^2$
   (c) $21xy^2 - 49x^2y$ (d) $\pi r^2 - \pi R$
   (e) $63a^2t - 45at^2 + 72at$ (f) $\frac{1}{2}R^2 - \frac{3}{2}RT$

4. Factor each:
   (a) $6x^2 - 54xy$
   (b) $3b(x + 2y) + 2c(x + 2y)$
   (c) $x(2a - 3c) + y(2a - 3c)$
   (d) $y^2(5a - 7d) - x^2(5a - 7d)$
   (e) $k(T^2 - t) + K(T^2 - t)$

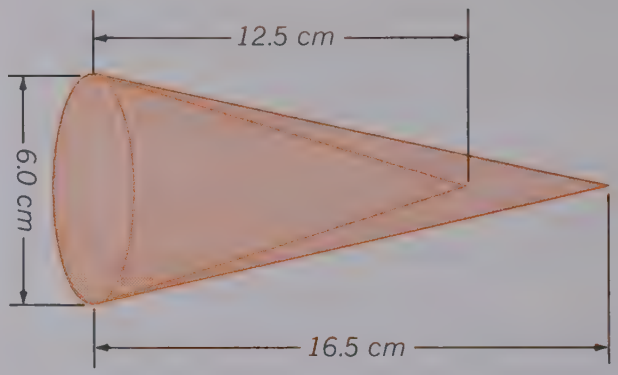
5. Calculate the area of each shaded ring:



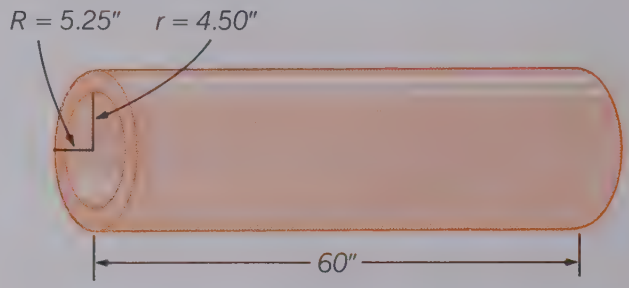
6. Calculate the area of each shaded region.



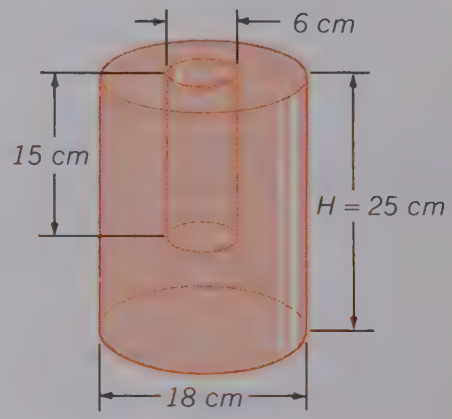
7. A machinist turns a partially hollowed cone as shown below. The hollow part is the shape of a cone. Find the volume of the remaining solid. (*Hint!* volume of cone $= \frac{1}{3}\pi r^2 h$).



8. Find the volume of the solid part of the hollow pipe shown below. (*Hint!* volume of cylinder $= \pi r^2 l$).



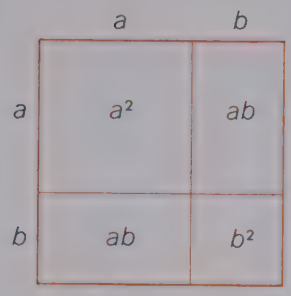
9. A lead container used to store radioactive materials is shown below. Find the volume of lead in the container. Lead weighs 11.4 g per cubic centimetre. What is the weight of the container as it stands?

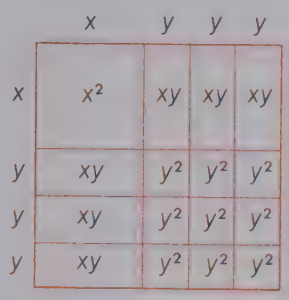


8·4 Squaring Binomials = Trinomials

Note the pattern in each:

(i) $(a + b)^2 = (a + b)(a + b)$
$= a^2 + 2ab + b^2$





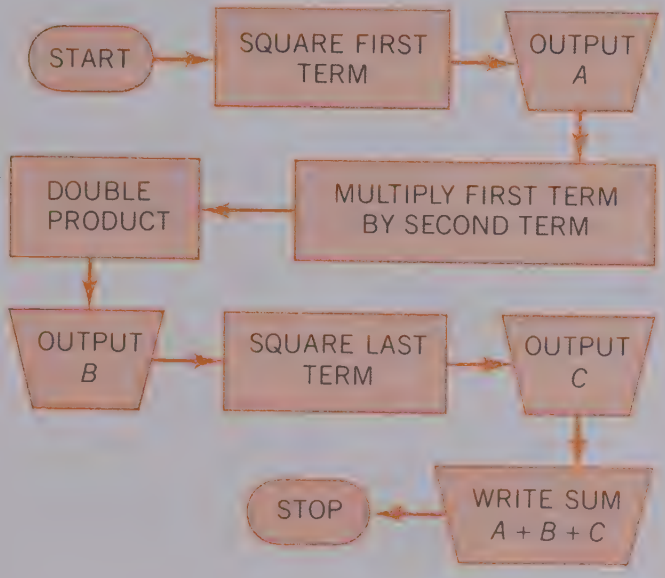
(ii) $(x + 3y)^2 = (x + 3y)(x + 3y)$
$= x^2 + 6xy + 9y^2$

(iii) $(2m - 3n)^2 = (2m - 3n)(2m - 3n)$
$= 4m^2 - 12mn + 9n^2$

(iv) $(ax + by)^2 = (ax + by)(ax + by)$
$= a^2x^2 + 2abxy + b^2y^2$

The pattern that is established here is illustrated in this flow chart:

To Square a Binomial



Example 1

Expand $(2a - 3b)^2$

Solution

Using the flow chart:

Square first term: $(2a)^2 = 4a^2$ ← A

Multiply first term by second term and double:

$(2a)(-3b) = -6ab$

$2(-6ab) = -12ab$ ← B

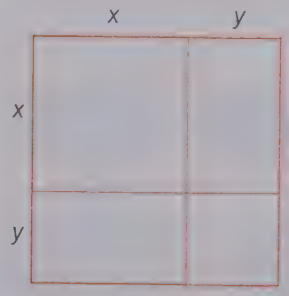
Square last term: $(-3b)^2 = 9b^2$ ← C

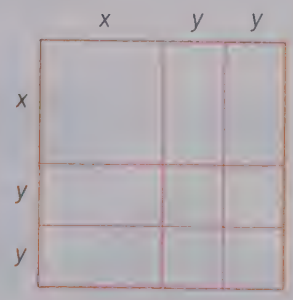
Sum of $A + B + C$:

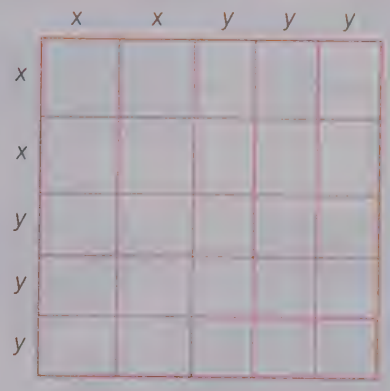
$4a^2 + (-12ab) + 9b^2 = 4a^2 - 12ab + 9b^2$

EXERCISE 8·4

1. Find the area of each square. Write each as a binomial squared.







2. Expand each:
   (a) $(d + c)^2$
   (b) $(s + 2t)^2$
   (c) $(s - 2t)^2$
   (d) $(3r + 2t)^2$
   (e) $(3r - 2t)^2$
   (f) $(4x + 3y)^2$.

8·5 Factoring a Trinomial Square

As the name indicates a trinomial square is a trinomial that is the perfect square of a binomial. We reverse the procedure of the preceding section.

(i) $a^2 + 2ab + b^2 = (a + b)^2$

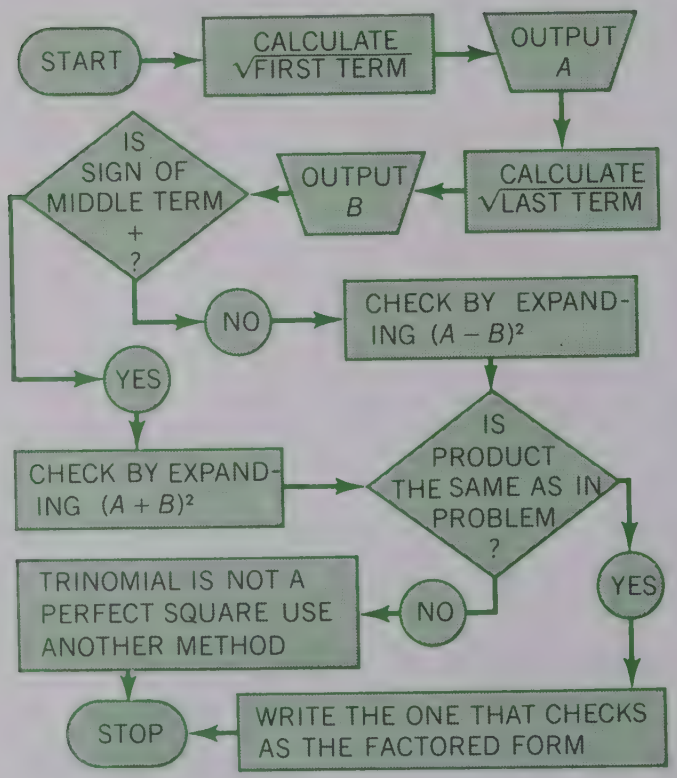
?	?
a^2	ab
ab	b^2

?	?	?	?
x^2	xy	xy	xy
xy	y^2	y^2	y^2
xy	y^2	y^2	y^2
xy	y^2	y^2	y^2

(ii) $x^2 + 6xy + 9y^2 = (x + 3y)^2$

(iii) $4m^2 - 12mn + 9n^2 = (2m - 3n)^2$

This flow chart will assist you in factoring a trinomial square.



Example 1

Factor each:

(a) $25x^2 - 60xy + 36y^2$

(b) $16x^2 + 50xy + 25y^2$

Solution:

(a) Calculate: $\sqrt{25x^2} = 5x \longleftarrow A$

$\sqrt{36y^2} = 6y \longleftarrow B$

Sign of middle term is not +.

Therefore expand $(A - B)^2$:

$(5x - 6y)^2 = 25x^2 - 60xy + 36y^2$

Does it check? Yes.

Therefore $(5x - 6y)^2$ is the factored form.

$25x^2 - 60xy - 36y^2 = (5x - 6y)^2$

(b) Calculate: $\sqrt{16x^2} = 4x \longleftarrow A$

$\sqrt{25y^2} = 5y \longleftarrow B$

Sign of middle term is +.

Check: $(A + B)^2$

$(4x + 5y)^2 = 16x^2 + 40xy + 25y^2$

Does it check? *NO!*

Therefore the trinomial is not a perfect square. At this time we do not have another method.

Therefore we shall leave it unfactored.

EXERCISE 8·5

1. Determine the dimensions of each square.

s^2	s^2	s^2	sr	sr
s^2	s^2	s^2	sr	sr
s^2	s^2	s^2	sr	sr
sr	sr	sr	r^2	r^2
sr	sr	sr	r^2	r^2

t^2	t^2	tn
t^2	t^2	tn
tn	tn	n^2

2. Factor each trinomial square that you can.

(a) $4x^2 - 4x + 1$ (b) $9x^2 + 30x + 25$

(c) $4x^2 + 12rs + 9s^2$ (d) $9m^2 - 30mn + 25n^2$

(e) $36a^2 - 84ab + 49b^2$ (f) $25a^2 - 75ab + 36b^2$

(g) $81k^2 - 90kl + 25l^2$ (h) $49m^2 - 56mn + 16n^2$

3. Supply the missing terms so that each will be a perfect square.

(a) $4m^2 + \blacksquare + 25n^2$	(b) $m^2 - \blacksquare + n^2$
(c) $9x^2 - \blacksquare + 16y^2$	(d) $49k^2 - \blacksquare + 36l^2$
(e) $9x^2 + \blacksquare + 16y^2$	(f) $m^2 + \blacksquare + n^2$
(g) $\blacksquare + 12xy + 36x^2$	(h) $\blacksquare - 12xy + 36x^2$
(i) $\blacksquare + 56ab + 49b^2$	(j) $\blacksquare - 36wv + 81v^2$
(k) $81m^2 - 18mn + \blacksquare$	(l) $81m^2 + 18mn + \blacksquare$
(m) $100k^2 - 20kl + \blacksquare$	(n) $25m^2 + 60mn + \blacksquare$

4. Weigh an empty pop (soft drink) can. Check with the person in charge of the dispenser in or near your school to determine how many cans of pop are sold per week in your school. Assume this number is an average for each of the schools in your district. What is the weight (grams? pounds? tons?) of the metal in the form of pop cans "wasted" in your district per year?

8·6 The Difference of Two Squares

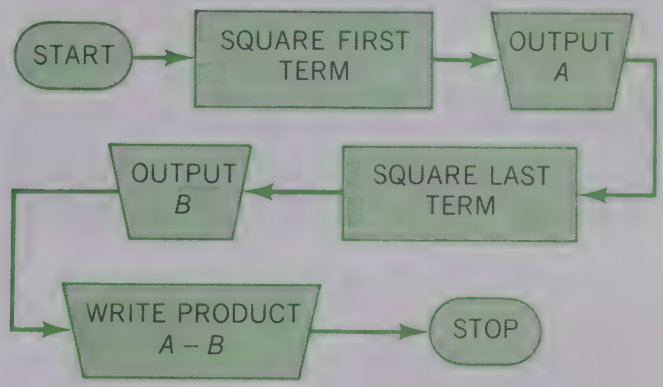
Note the pattern established in these products:

(i) $(a + b)(a - b) = a^2 - ab + ab - b^2$
$= a^2 - b^2$

(ii) $(a - 3b)(a + 3b) = a^2 + 3ab - 3ab - 9b^2$
$= a^2 - 9b^2$

(iii) $(2x - 5y)(2x + 5y) = 4x^2 + 10xy - 10xy - 25y^2$
$= 4x^2 - 25y^2$

The product of these binomials are binomials. They differ from the previous products in that the *middle term* is missing. Why are the *middle terms* missing? The pattern established for the product of the sum and difference of the same two terms is represented in this flow chart.



Example 1

Find the product: $(5a + 4b)(5a - 4b)$

Solution

Note this is the product of the sum and difference of the same two terms. Hence we will use the flow chart above:

Square first term: $(5a)^2 = 25a^2 \longleftarrow A$
Square second term: $(4b)^2 = 16b^2 \longleftarrow B$
Product: $25a^2 - 16b^2$
Therefore: $(5a + 4b)(5a - 4b) = 25a^2 - 16b^2$
We can use this method of multiplying to help us.

Example 2

Calculate the product 49×51.

Solution

49×51 may be written $(50 - 1)(50 + 1)$.
Using the method illustrated we can multiply easily:

$$\begin{aligned} 49 \times 51 &= (50 - 1)(50 + 1) \\ &= 50^2 - 1^2 \\ &= 2500 - 1 \\ &= 2499 \end{aligned}$$

Use conventional multiplication method to check this.

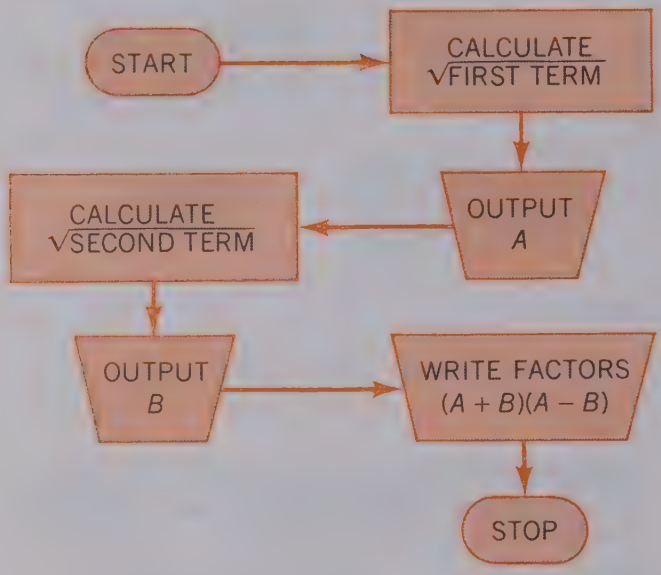
Do questions 1 through [illegible]

Let us now study the factoring of the *difference of two squares*.

(i) $a^2 - b^2 = (a + b)(a - b)$
(ii) $a^2 - 9b^2 = (a + 3b)(a - 3b)$
(iii) $4x^2 - 25y^2 = (2x + 5y)(2x - 5y)$

After studying the factors of each, state in your own words how to factor the *difference of two squares*.

This flow chart presents one approach to the factoring of the difference of two squares.



It should be noted that the student must establish that the problem is the *difference of two squares*. If it is not, then this method cannot be used.

Example 3

Factor: $49x^2 - 36y^2$

Solution

This is the difference (–) of two squares.

$$\sqrt{\text{first term}} = \sqrt{49x^2}$$
$$= 7x \longleftarrow A$$

$$\sqrt{\text{second term}} = \sqrt{36y^2}$$
$$= 6y \longleftarrow B$$

Factors: $(A + B)(A - B) = (7x + 6y)(7x - 6y)$

Therefore: $49x^2 - 36y^2 = (7x + 6y)(7x - 6y)$

EXERCISE 8·6

Read at sight, the product in each case.

1. $(a - 2b)(a + 2b)$
2. $(x + 4y)(x - 4y)$
3. $(7y + 1)(7y - 1)$
4. $(1 + 7k)(1 - 7k)$
5. $(ab + 5)(ab - 5)$
6. $(11 - n)(11 + n)$
7. $(a^2 - 2)(a^2 + 2)$
8. $(4 - xy)(4 + xy)$
9. $(5x - 2y)(5x + 2y)$
10. $(3a + k)(3a - k)$
11. $(x^3 - y)(x^3 + y)$
12. $(6x - 5y)(6x + 5y)$

Use the product of the sum and difference of two numbers to help you make each calculation. (See Example 2.)

13. 48×52
14. 39×41
15. 99×101
16. 69×71
17. 999×1001
18. 76×84
19. State in your own words how you can identify the *difference of squares* factoring problem.

Factor each completely if possible.

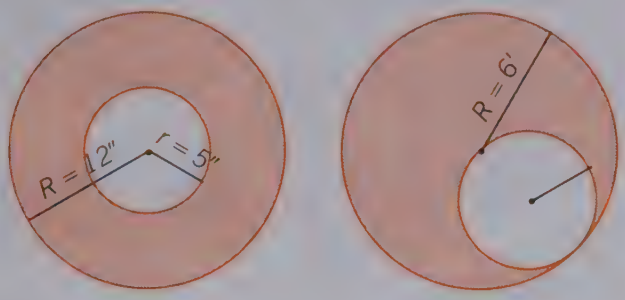
20. $c^2 - d^2$
21. $m^2 - n^2$
22. $36y^2 - 25x^2$
23. $100s^2 - 81d^2$
24. $9r^2 + 16s^2$
25. $\frac{1}{4}x^2 - \frac{1}{9}y^2$
26. $\frac{9}{4}y^2 - \frac{25}{49}x^2$
27. $\frac{1}{9}x^2 - \frac{1}{16}y^2$
28. $144 + t^2$
29. $R^2 - 100$
30. $T^2 - 121$
31. $1 - 64a^2$

Factor each fully. Remember to factor out the common factor before attempting other types of factoring.

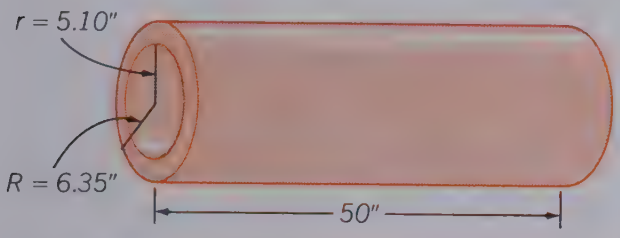
32. $a^4 - b^4$
33. $8y^2 - 18x^2$
34. $c^4 - d^4$
35. $d^4 - 1$
36. $4w^2 - 4$
37. $8x^2 - 32y^2$

38. Calculate the area of each ring in question 5 of Exercise 8·3 by using the factors of the difference of squares to simplify the multiplications.

39. Calculate the area of each:



40. Calculate the volume of the solid part of this pipe:



41. The length of each sewer pipe below is 8 feet. The inside radius is 36" and the outside radius is 42". What is the volume of concrete in the two pipes?

42. Find the weight and volume of all the telephone directories discarded this year in your telephone directory area. If scrap paper is worth $0.30 per hundred pounds what is the value of the discarded directories?

What is your opinion about the dumping of telephone directories in the garbage?

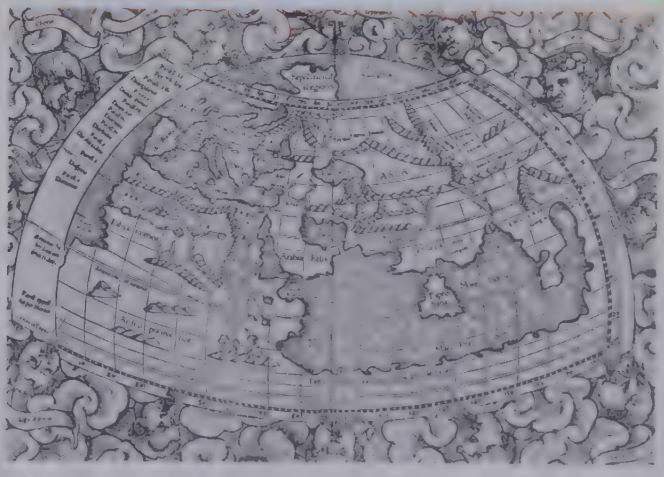
What do you think should be done with old directories?

Using the Library

The Wonderful World of Mathematics: Read pages 58, 59 and 60.

Life Science Library Mathematics: Read pages 80-88.

Write a paragraph on René Descartes and his life.



It was in the 15th century that map makers first used grids with lines of latitude and longitude to draw maps. It was not until the 17th century that the same concept was used to locate points in a plane surface in mathematics.

REVIEW EXERCISE 8

1. Find the sum in each case:
   (a) $5a - 3b + 8b - 2a$
   (b) $27m - 12n + 15m - 19n - 50m$.

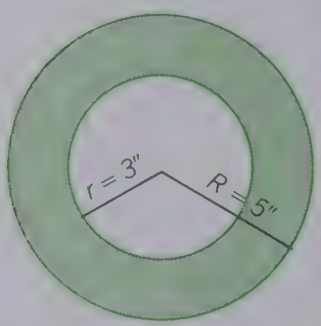
2. Perform the indicated operations and simplify.
   (a) $5k(12k - 7l)$
   (b) $(2a - 3m)(2a - 3m)$
   (c) $8a(2a^2 - 4a - 5)$
   (d) $6a - (2a - 3b) - 3b$
   (e) $\dfrac{35a^3 - 15a}{5a}$
   (f) $\dfrac{120x^3 - 96x^2 - 16x}{16x}$.

3. Multiply and check:
   (a) $(3a - 2b)(5a - 6b)$
   (b) $(7x - 2y)(2x + 3y)$
   (c) $(3a - 5b)^2$
   (d) $(2x - 3y)(2x + 3y)$.

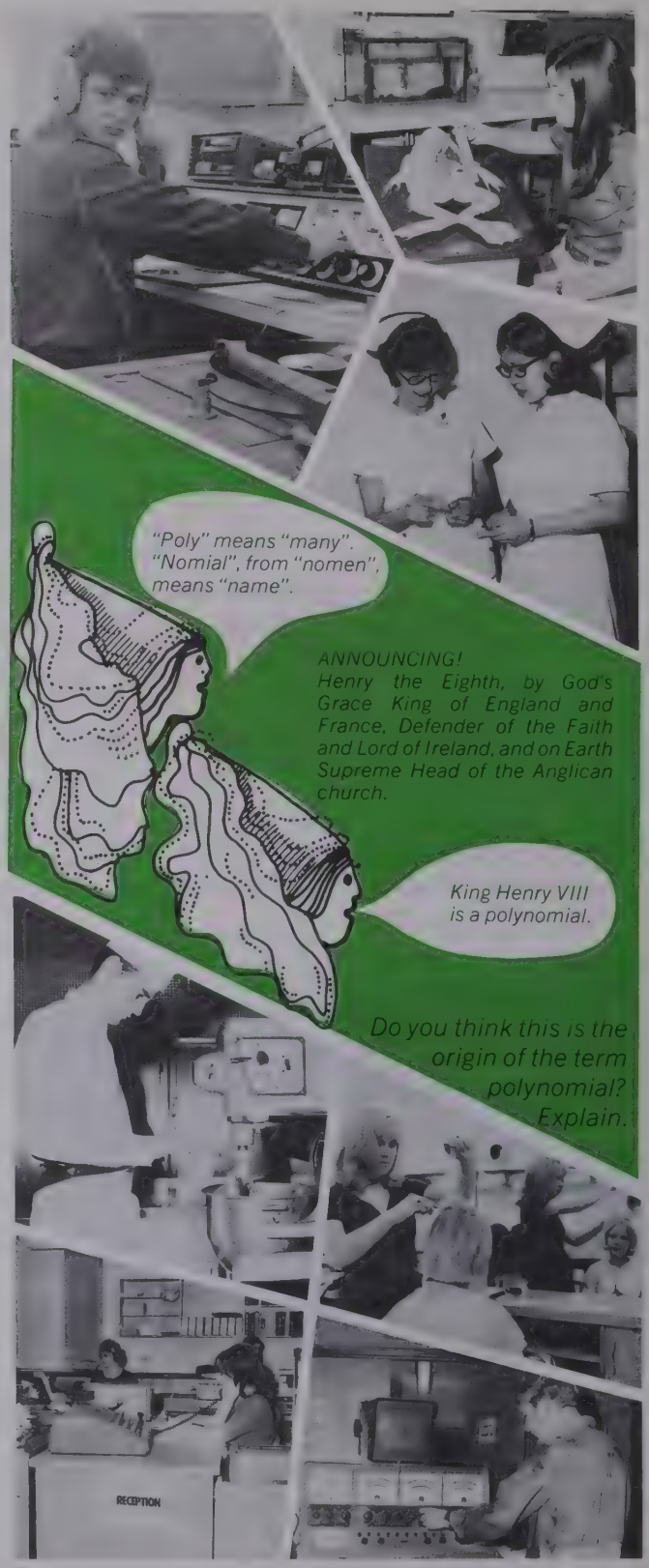
4. Factor each:
   (a) $3m^2n - 6mn^2$
   (b) $4v^2 - 4vw + w^2$
   (c) $9m^2 + 30mn + 25n^2$
   (d) $64 - 81y^2$.

5. Multiply: 52×48.

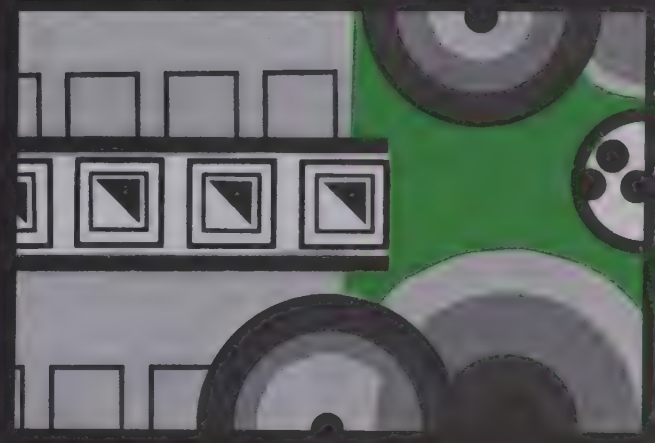
6. Calculate the area of the shaded portion.



Discuss how each person pictured on this page uses math.


"Poly" means "many". "Nomial", from "nomen", means "name".
ANNOUNCING! Henry the Eighth, by God's Grace King of England and France, Defender of the Faith and Lord of Ireland, and on Earth Supreme Head of the Anglican church.
King Henry VIII is a polynomial.
Do you think this is the origin of the term polynomial? Explain.
RECEPTION

Chapter Nine
GEOMETRY



9·1 Introduction

Geometry, in its infancy, developed as the needs of civilizations demanded. Possibly it began when man wished to measure land. Indeed, the word geometry comes from the Greek words meaning measurement (meter) of the earth (geo-). Geometry became a science in the Nile Valley of ancient Egypt since each year the river overflowed its channel onto the flat river valley plains and washed away the boundaries that marked each landowner's property. The land had to be surveyed after each flood to re-establish the boundary lines. In addition, overseers had to determine, for taxation purposes, the size of each parcel of land. While the Egyptians were achieving great precision in measurement and in construction – the Great Pyramid has a square base of length 755.8 feet and the four lengths differ by less than 0.7 inches – other parts of the then known world were making advances in geometry. Along the Tigris and Euphrates of Mesopotamia, the Indus and Ganges of southern Asia, and the Yangtze and Hwang Ho of China, developed civilizations which necessitated an understanding of and a practical knowledge of geometry. These sites of ancient civilizations bear, to this date, evidence that a practical geometry had evolved.

There are many names associated with the various aspects of geometry. While Pythagoras is perhaps the most well-known, most high school geometry is based on the extensive but comprehensive work (13 books) by Euclid entitled *Elements*. This collection of 465 propositions produced about 300 B.C. was Euclid's compilation of the knowledge in geometry to that date. The student should look up a few references to learn of other mathematicians and their accomplishments.

Things are not always as they appear. We must always be critical of impressions we obtain by means of the senses. In geometry we must be critical of what we think and of what we think we see. We must have some foundation, some understanding upon which we can rely and upon which we can base our critical thinking. The projects in this unit are designed to increase your ability to think critically and to draw conclusions from the facts presented.

In this section we shall study some equalities of measures of geometric figures and the resulting relations between the geometric figures.

Let us begin by considering the two segments shown in Figure 9-1.

Euclid. Greek mathematician.

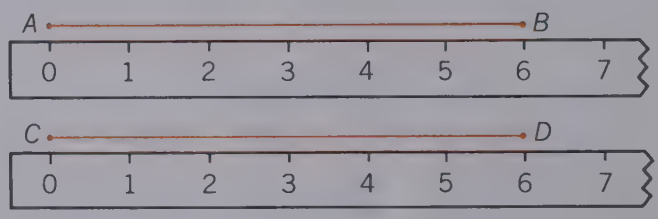


Figure 9-1

We see the measure of $\overline{AB}$ is the same as the measure of $\overline{CD}$. The measures are equal (since the measures are numbers they can be equal) so we write

$$m(\overline{AB}) = m(\overline{CD}).$$

Another way of conveying the same idea is to say the two segments are *congruent*.

$$\overline{AB} \equiv \overline{CD}.$$

This is read "Segment *AB* is congruent to segment *CD*."

Similarly we say the measure of $\angle ABC$ is equal to the measure of $\angle DEF$ so we write

$$m\angle ABC = m\angle DEF.$$

When the measures of two angles are equal we say the angles are congruent. Hence in the case in Figure 9-2 we write

$$\angle ABC \equiv \angle DEF$$

How do you read this?

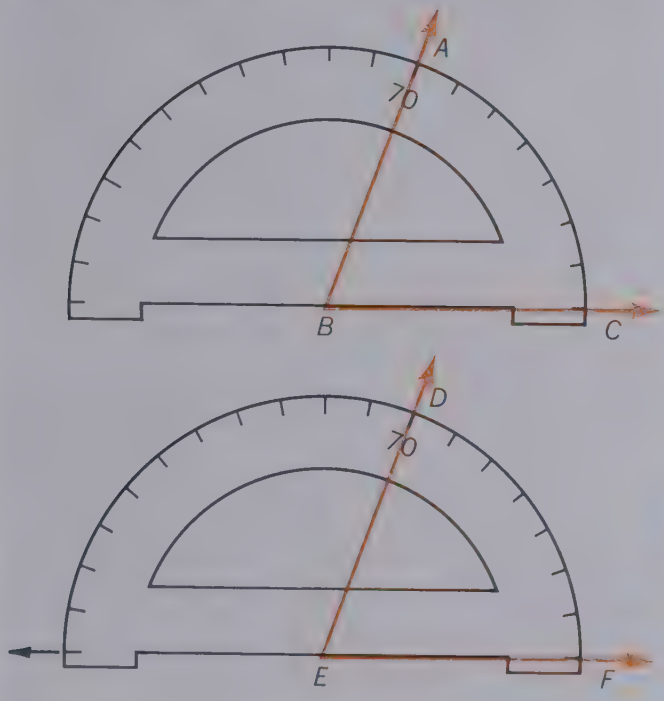


Figure 9-2

Refer to Figure 9-3. Use a ruler to find the measure in centimetres of each segment.

$\begin{cases} \overline{MN} = \blacksquare \\ \overline{PQ} = \blacksquare \end{cases}$ $\begin{cases} \overline{NO} = \blacksquare \\ \overline{QR} = \blacksquare \end{cases}$ $\begin{cases} \overline{MO} = \blacksquare \\ \overline{PR} = \blacksquare \end{cases}$

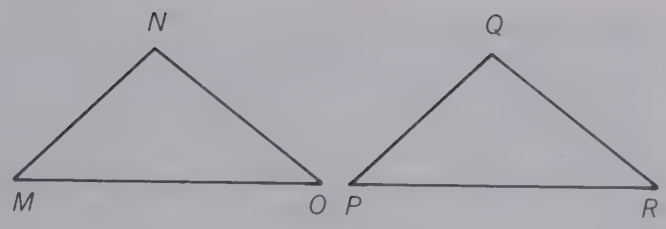


Figure 9-3

Use a protractor to find the measure in degrees of each angle.

$\begin{cases} \angle M = \blacksquare \\ \angle P = \blacksquare \end{cases}$ $\begin{cases} \angle N = \blacksquare \\ \angle Q = \blacksquare \end{cases}$ $\begin{cases} \angle O = \blacksquare \\ \angle R = \blacksquare \end{cases}$

Did you find the measures of each pair to be equal? Notice that $\overline{MO}$ is the longest side of $\triangle MNO$ and $\overline{PR}$ is the longest side of $\triangle PQR$. We say $\overline{MO}$ and $\overline{PR}$ are *corresponding sides of the triangles*. Also $\angle N$ is the largest angle of $\triangle MNO$ and $\angle Q$ is the largest angle of $\triangle PQR$. $\angle N$ and $\angle Q$ are *corresponding angles of the triangles*. Name the other corresponding sides and the other corresponding angles of these triangles. Since three corresponding sides and three corresponding angles of the two triangles are congruent we say the two triangles are congruent. Also, the converse of this statement holds: if the two triangles are congruent then we know the corresponding sides and the corresponding angles are congruent. In general, two geometric figures are congruent if they have the same size and the same shape.

Which 2 shapes are congruent?

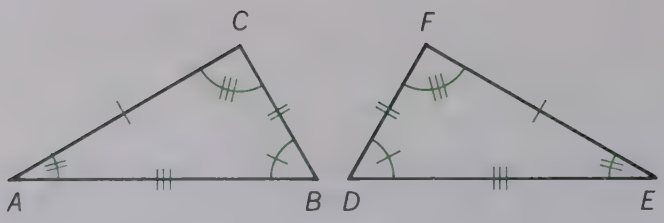


Figure 9-4

In Figure 9-4, the slashes tell us which parts of the triangles are congruent. For example, $\overline{AC} \equiv \overline{FE}$ (they each have one slash) and $\angle B \equiv \angle D$. List the congruent parts. Are the congruent parts corresponding parts? We write

$$\triangle ACB \equiv FED.$$

Note how this is written:

$$\triangle A\,C\,B \equiv \triangle F\,E\,D$$

$\angle A$ corresponds to $\angle F$.
$\angle C$ corresponds to $\angle E$.
$\angle B$ corresponds to $\angle D$.

$$\triangle A\,C\,B \equiv \triangle F\,E\,D$$

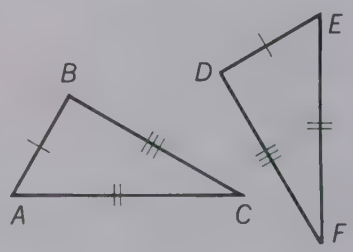
$\overline{AC}$ corresponds to $\overline{FE}$.
$\overline{CB}$ corresponds to $\overline{ED}$.
$\overline{AB}$ corresponds to $\overline{FD}$.

The names of the triangles are written in such a manner that the order of the correspondences is the same for both triangles.

Use the above procedure to decide whether or not we could write the following for Figure 9-4:
$\triangle CBA \equiv \triangle DFE$ $\triangle BAC \equiv \triangle EFD$ $\triangle BCA \equiv \triangle DEF$

EXERCISE 9·1

1. Given the two congruent triangles which statements are correct?



$\triangle ABC \equiv \triangle DEF$
$\triangle ABC \equiv \triangle FED$
$\triangle ABC \equiv \triangle EDF$
$\triangle BAC \equiv \triangle DEF$
$\triangle BCA \equiv \triangle EDF$
$\triangle CBA \equiv \triangle FDE$
$\triangle CAB \equiv \triangle FDE$

9·2 Rules of Congruency

To show that two triangles are congruent we need not show that all six corresponding parts of the triangles are congruent. We need to know that only three corresponding parts of the triangles are congruent provided they are the right three.

Projects

Project 1

Purpose

To explore conditions for congruency of triangles.

Method

A. 1. Construct a triangle ABC with $m(\overline{AB}) = 4$cm, $m(\overline{BC}) = 5$ cm, $m(\overline{CA}) = 6$ cm. Carefully cut out the triangle.

2. Compare your triangle with those of your classmates. Can you arrange all the triangles so that corresponding angles and corresponding sides each coincide?

3. Were there two or more distinct triangles constructed?

4. What three things were given for this triangle? Write a rule for congruence of triangles.

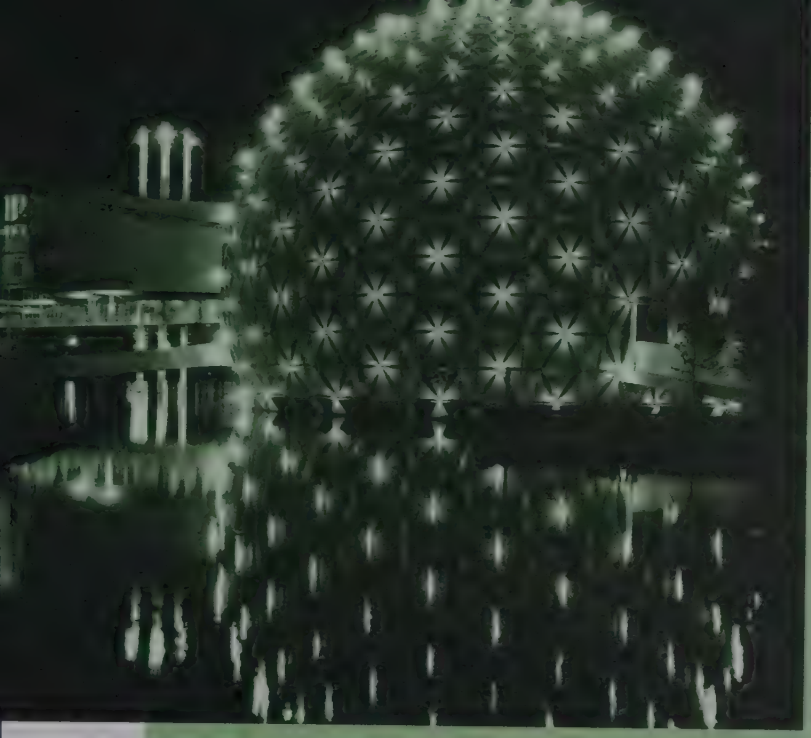
B. 1. Construct a triangle MNP with $m(\overline{MN}) = 3$ cm, $m(\overline{NP}) = 6$ cm, $m \angle MNP = 70°$. Notice $\angle MNP$ is the included angle. What do we mean by *included*?

2. Do steps 2, 3, and 4 of Part A.

C. 1. Construct a triangle XYZ with $m \angle X = 30°$, $m \angle Y = 70°$, and $m(\overline{XY}) = 6$ cm.

2. Do steps 2, 3 and 4 of Part A.
STOP! Do not proceed until you have completed Parts A, B, and C.

D. Check your rules with the three rules which follow.



Cinesphere, Ontario Place. Note the structure of congruent triangles.

Rule 1

If three sides of a triangle are congruent to the corresponding sides of a second triangle, then the two triangles are congruent. (*S.S.S.*)

Rule 2

If two sides and the included angle of one triangle are congruent to the corresponding parts of a second triangle, then the two triangles are congruent. (*S.A.S.*)

Rule 3

If two angles and a side of one triangle are congruent to the corresponding parts of a second triangle, then the two triangles are congruent. (*A.A.S.*)

Draw pairs of triangles to illustrate each rule. Label the given congruent parts in each pair.

Project 2

Purpose

To study triangles in reference to congruency.

Method

1. Construct triangle ABC in which $m(\overline{AB}) = 5$ cm, $m(\overline{BC}) = 7$ cm, $m \angle C = 40°$.

2. Compare your triangle with those of your classmates. Are all the triangles congruent? How many different triangles can you draw with the information given?

3. A student endeavored to draw $\triangle FGH$ given $m \angle H = 60°$, $m(\overline{FG}) = 2$ cm, $m(\overline{GH}) = 6$ cm. He could not draw the triangle. Can you help him or explain the situation?

4. Draw two different triangles which are not congruent given $m \angle R = 30°$, $m(\overline{TS}) = 4$ cm, $m(\overline{SR}) = 6$ cm.

5. In parts 1, 3, 4 above there have been two sides and an angle given in each case. In part 3 it is impossible to draw such a triangle. In each of 1 and 4 it is possible to draw two triangles which are not congruent yet which have two corresponding sides and a corresponding angle congruent. Explain.

6. Is it sufficient to have two sides and an angle of one triangle congruent to the corresponding parts of another triangle to show the two triangles congruent? What condition regarding the angle is necessary? *Hint!* Consider Rule 2.

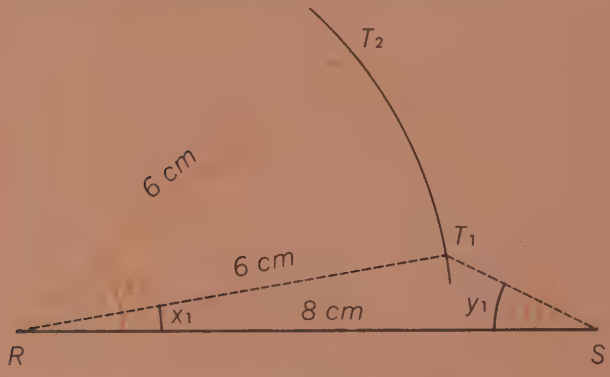
Project 3

Purpose

To explore conditions necessary for congruency.

Method

1. Draw $\overline{RS}$ with measure 8 cm.
   With centre R draw an arc as shown with radius of measure 6 cm.
   Note, for any point T in the arc
   $$m(\overline{RT}) = 6 \text{ cm}.$$
   Let x be the measure of $\angle TRS$ in degrees.
   Let y be the measure of $\angle TSR$ in degrees.



2. Draw $\overline{RT}$ such that $x = 10°$. Draw $\overline{ST}$.
   Measure y. Then repeat for $x = 20°, 30°, \ldots, 160°$.
   Record the information in the table.

x	10°	20°	30°	. . .	160°
y					

   Plot the ordered pairs on a graph and draw a smooth curve through the points. Is this a graph of a function?

3. Record the information in the table from the above experiment.

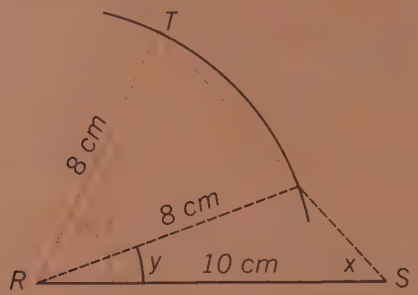
x	10°	20°	30°	. . .	160°
$m(\overline{TS})$					

   Plot the ordered pairs on a graph. Let the usual y-axis be the axis for $m(\overline{TS})$. Draw a smooth curve through the points. Is this a graph of a function?

4. From 2 above, for any given x, is there more than one value for y?

5. From 3 above, for any given x, is there more than one value for $m(\overline{TS})$?

6. Repeat the procedure above for this situation. Note the angles are named differently.
   $m(\overline{RS}) = 10$ cm and $m(\overline{RT}) = 8$ cm.



x	10°	20°	30°	40°	50°
y					

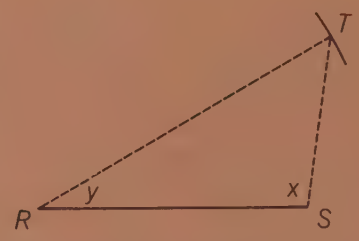
   Draw the graph. Is this a graph of a function? From the graph, for any given value of x is there more than one value of y?

7. If you were given $m(\overline{RT}) = 6$ cm, $m(\overline{RS}) = 8$ cm and $m \angle S = 40°$ how many different triangles can you draw?

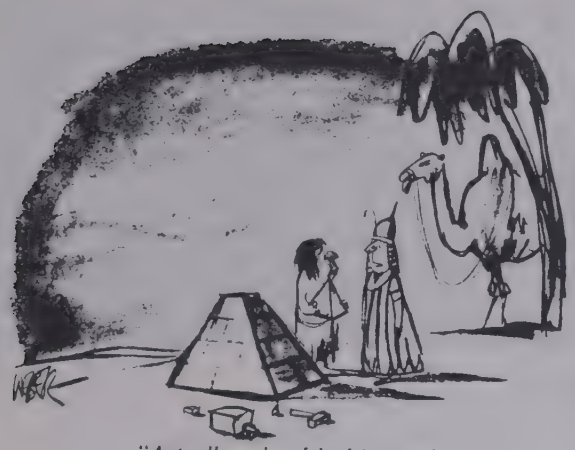
8. Do two sides and an angle *not included* always determine a triangle?

9. Repeat the above procedure given $m(\overline{RS}) = 8$ cm, and $m(\overline{RT}) = 10$ cm.

x	10°	20°	30°	. . .	160°
y					



   Draw the graph. Is this a graph of a function? Do two sides and an angle *not included* determine a triangle in this case?



"Actually, what I had in mind was something a little larger."

Drawing by Weber; © 1963 The New Yorker Magazine, Inc.

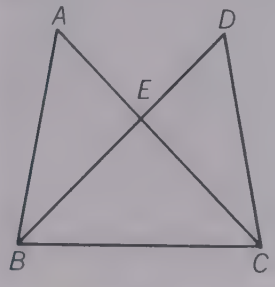
9·3 Using Rules of Congruency

We will use the three rules for congruent triangles along with other rules you have learned earlier to show triangles are congruent.

Example 1

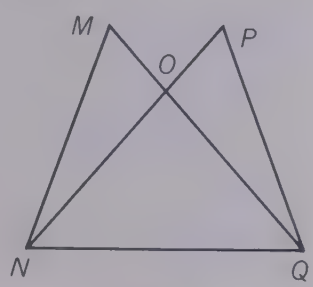
Label with slash marks the corresponding congruent parts of the triangles listed. Then determine whether or not the two triangles are congruent. If they are congruent state which rule applies.

(a)



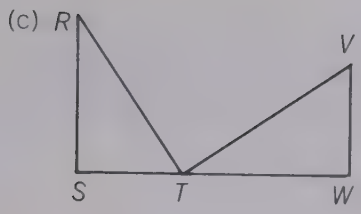
$\overline{AB} \equiv \overline{DC}$
$\angle ABC \equiv \angle DCB$
$\triangle ABC \; ? \; \triangle DCB$

(b)



$\angle M \equiv \angle P$
$\overline{MN} \equiv \overline{PQ}$
$\triangle MNO \; ? \; \triangle PQO$

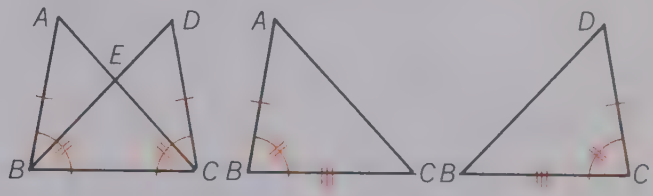
(c)



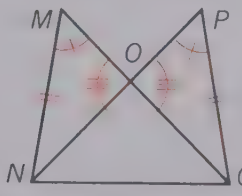
$m\angle S = 90°$
$m\angle W = 90°$
$\angle TRS \equiv \angle VTW$
$\overline{RT} \equiv \overline{TV}$
$\triangle RST \; ? \; \triangle TWV$

Solution

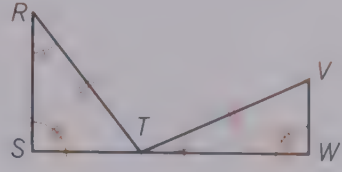
(a) After placing the slash marks on, it is sometimes desirable to draw the two triangles separately. We now are able to see clearly that the side $\overline{BC}$ is congruent to $\overline{BC}$. This is often called a *common side* since it is in both triangles. Hence, we mark this side with (three) slash marks. We now have 2 sides and the included angle of $\triangle ABC$ congruent to the corresponding parts of $\triangle DCB$. Therefore $\triangle ABC \equiv \triangle DCB$ by the S.A.S. Rule.



(b) Again place slash marks on the diagram from the given information. Further we know $\angle MON \equiv \angle POQ$ since they are vertically opposite angles. Mark these angles with appropriate slash marks. We now have two angles and a side of $\triangle MNO$ congruent to the corresponding parts of $\triangle PQO$. Therefore $\triangle MNO \equiv \triangle PQO$ by A.A.S. Rule.



(c) Since the measure of angle S is equal to the measure of angle W, $\angle S \equiv \angle W$. We have two angles and a side of $\triangle RST$ congruent to corresponding parts of $\triangle TWV$. Therefore, $\triangle RST = \triangle TWV$ by the A.A.S. Rule.

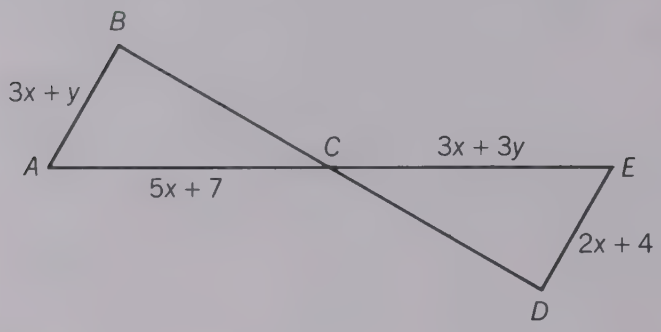


Example 2

Find, if possible, the dimensions required in the triangles given. The measures of the sides are given on the triangles.

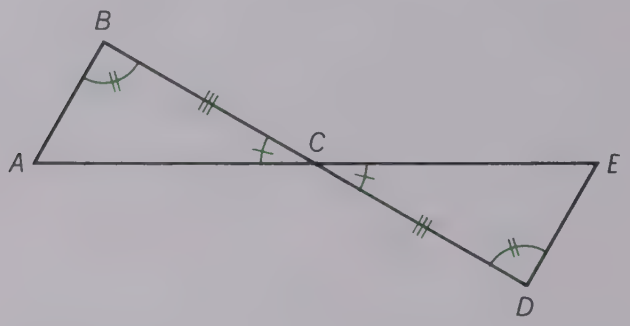
Given: $\angle ABC \equiv \angle EDC$
$\overline{BC} \equiv \overline{DC}$

Required: $m(\overline{AB}) = \blacksquare$
$m(\overline{AC}) = \blacksquare$



Solution

Since the corresponding parts marked are congruent, the two triangles are congruent by the A.A.S. Rule.

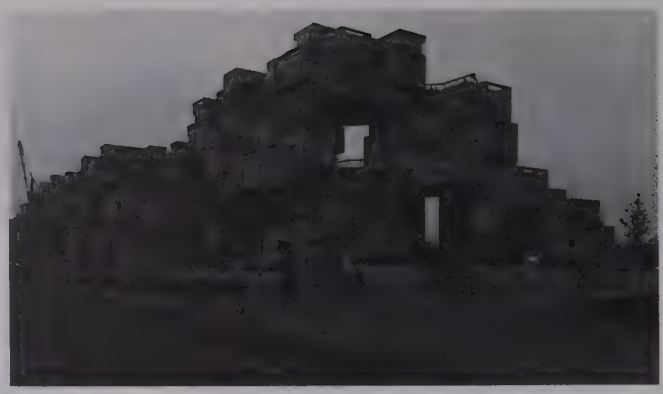


Hence $m(\overline{AB}) = m(\overline{DE})$ Why?
$\therefore$ ① $3x + y = 2x + 4$
and $m(\overline{AC}) = m(\overline{EC})$ Why?
$\therefore$ ② $5x + 7 = 3x + 3y$

Solving this system of linear equations in two variables we get $x = 1, y = 3$.

Since $m(\overline{AB}) = 3x + y$
$m(\overline{AB}) = 3 \cdot 1 + 3$
$m(AB) = 6.$

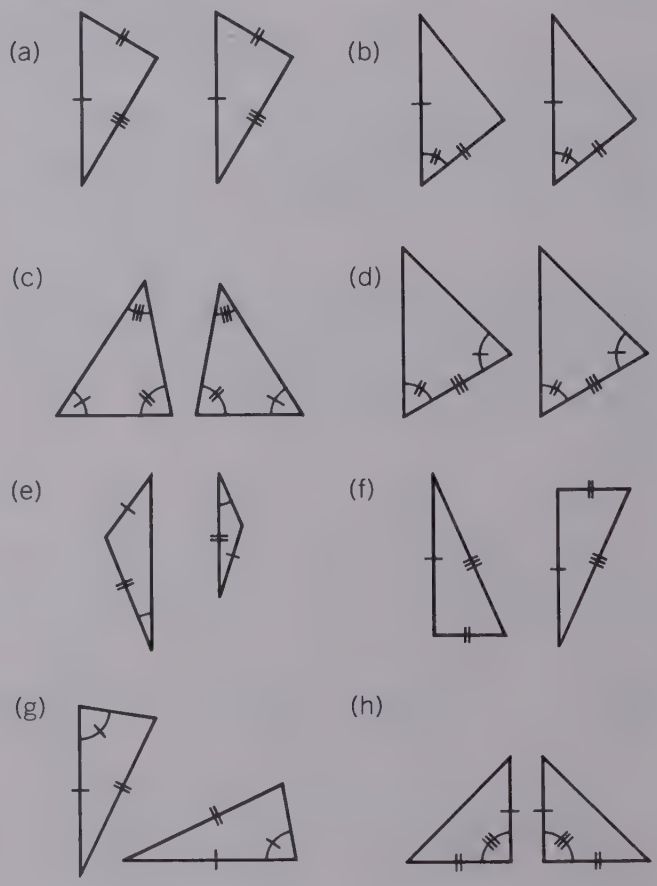
Similarly $m(\overline{AC}) = 5x + 7$
$m(\overline{AC}) = 12.$



Habitat, Montreal. Repetition of congruent shapes is utilized in modern architecture. The repetition ties parts of the building together to give the feeling of unity.

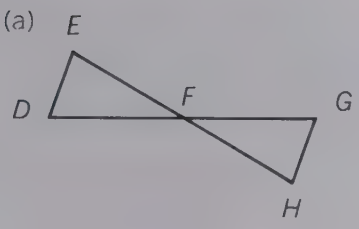
EXERCISE 9·3

1. Tell which pairs of triangles are congruent. Tell which rule applies to each pair of congruent triangles.



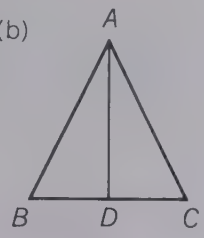
2. Label with slash marks the corresponding parts of the triangles shown. Then determine whether or not the two triangles are congruent. If they are congruent state the rule that applies.

(a)



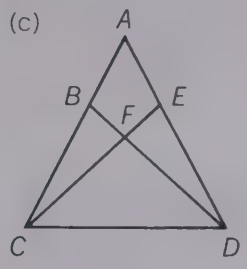
$\angle D \equiv \angle G$
$\overline{DF} \equiv \overline{GF}$
$\triangle DEF \; ? \; \triangle GHF$

(b)



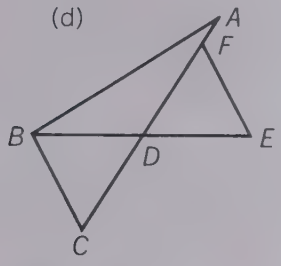
$\triangle ABC$ is isosceles
$\overline{BD} \equiv \overline{CD}$
$\triangle BDA \; ? \; \triangle CDA$

(c)



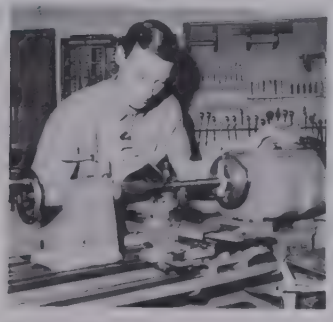
$\angle ACD \equiv \angle ADC$
$\overline{BC} \equiv \overline{ED}$
$\triangle BCD \; ? \; \triangle EDC$

(d)



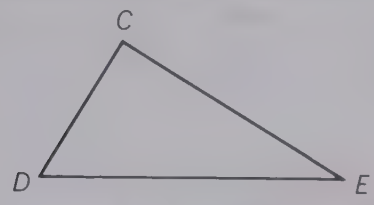
$\overline{BD} \equiv \overline{DF}$
$\overline{CD} \equiv \overline{DF}$
$\overline{FD} \equiv \overline{FE}$
$\angle DBC \equiv \angle DCB$
$\triangle BCD \; ? \; \triangle EDF$

3. Using compasses, ruler and protractor draw $\triangle MNO$ given $m\angle M = 40°$, $m(\overline{MN}) = 5$ cm, $m(\overline{MO}) = 6$ cm. Compare with your classmates. Are all the triangles in your class congruent? Why do you think they should or should not be?



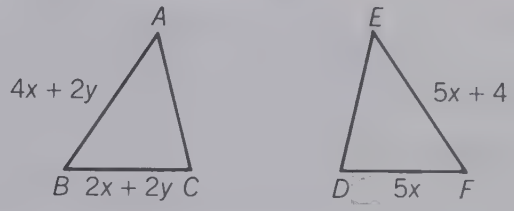
Discuss how a knowledge of geometry affects the work of these people. If you were an employer would you insist the employee have some knowledge of this topic?

4. Using compasses and straightedge only draw a triangle congruent to the given $\triangle CDE$.

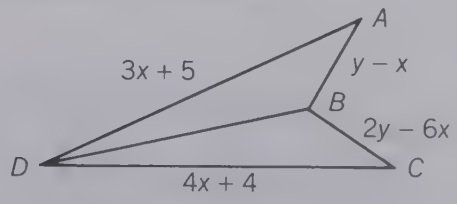


5. Find, if possible, the dimensions required in the triangles given. The measures of the two sides are given on the triangles.

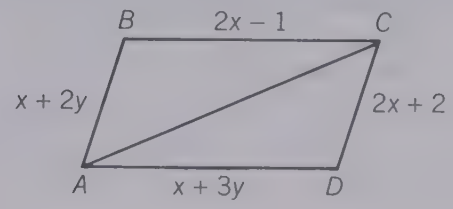
(a) *Given:* $\triangle ABC$ and $\triangle EDF$
$\overline{AC} \equiv \overline{ED}$
$\angle A \equiv \angle E$
$\angle B \equiv \angle F$
Required: $m(AB) =$ ■
$m(\overline{DF}) =$ ■



(b) *Given:* Figure *ABCD*.
$\angle A \equiv \angle C$
$\angle ADB \equiv \angle CDB$
Required: $m(AB) =$ ■



(c) *Given:* Figure *ABCD*.
$\angle ACB \equiv \angle CAD$
$\angle CAB \equiv \angle ACD$
Required: $m(AB) =$ ■
$m(AD) =$ ■



6. Obtain from the science laboratory, if possible, a model of a steam engine. A piston drives a rod which in turn drives a connecting rod attached to a wheel. As the piston is forced to the right the rods move causing the wheel to turn.

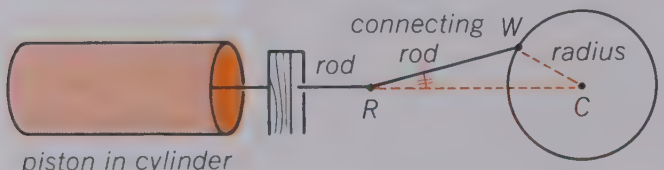


Diagram 1

Diagram 2

Does the length of the connecting rod $\overline{RW}$ change as the rod moves? Does the length of the radius $\overline{CW}$ change as the rod moves?

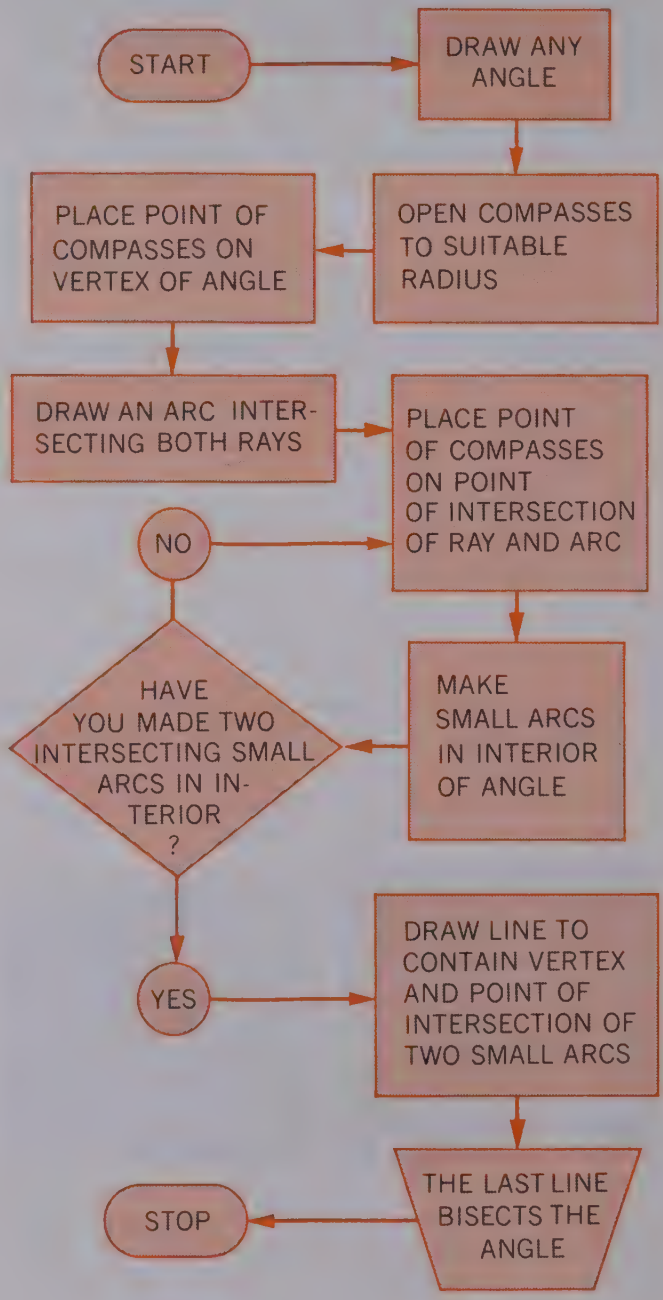
The measures of $\angle WRC$ and $\angle W_1R_1C$ are the same.
Is $\triangle WRC$ in diagram 1 congruent to $\triangle W_1R_1C$ in diagram 2?

If two sides and the angle opposite one of the sides of a triangle are congruent to corresponding parts of a second triangle are the two triangles necessarily congruent?

The steam engine was a big advancement over horse and oxen power. These engines were used to pull 24 bottom plows, run threshers, power sawmills and mining equipment. Can you find out what the horse-power *and speed of these giants were? What did the giants burn as fuel?*

9·4 Flow Chart Mathematics

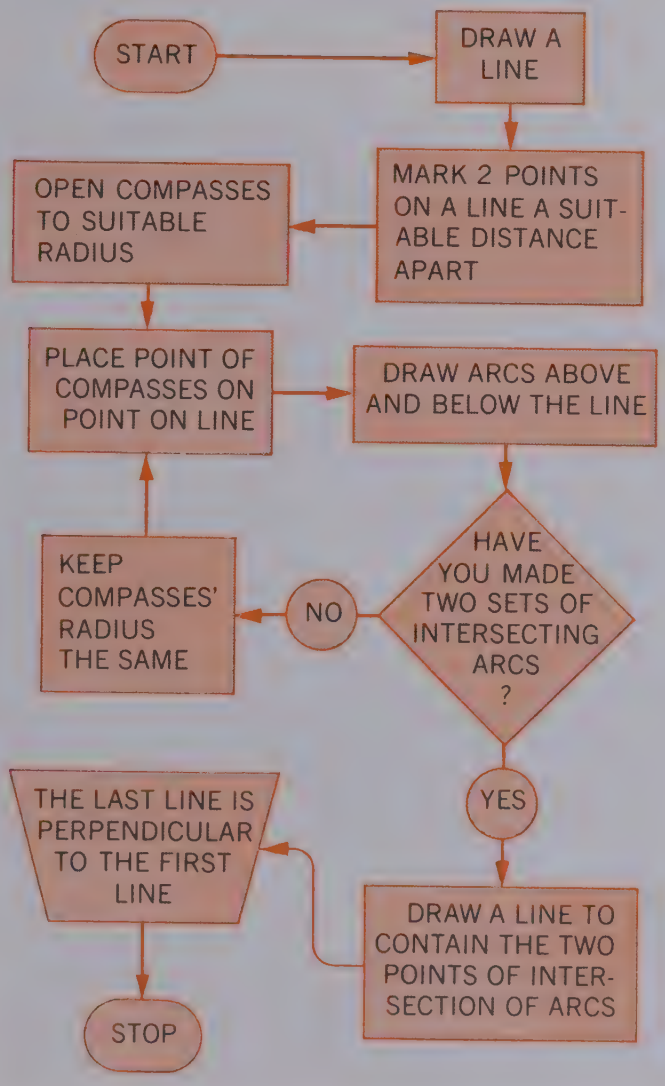
The following flow chart tells you how to bisect an angle. Follow the instructions in the chart. Can you?


START
DRAW ANY ANGLE
OPEN COMPASSES TO SUITABLE RADIUS
PLACE POINT OF COMPASSES ON VERTEX OF ANGLE
DRAW AN ARC INTERSECTING BOTH RAYS
PLACE POINT OF COMPASSES ON POINT OF INTERSECTION OF RAY AND ARC
NO
MAKE SMALL ARCS IN INTERIOR OF ANGLE
HAVE YOU MADE TWO INTERSECTING SMALL ARCS IN INTERIOR ?
YES
DRAW LINE TO CONTAIN VERTEX AND POINT OF INTERSECTION OF TWO SMALL ARCS
THE LAST LINE BISECTS THE ANGLE
STOP

EXERCISE 9·4

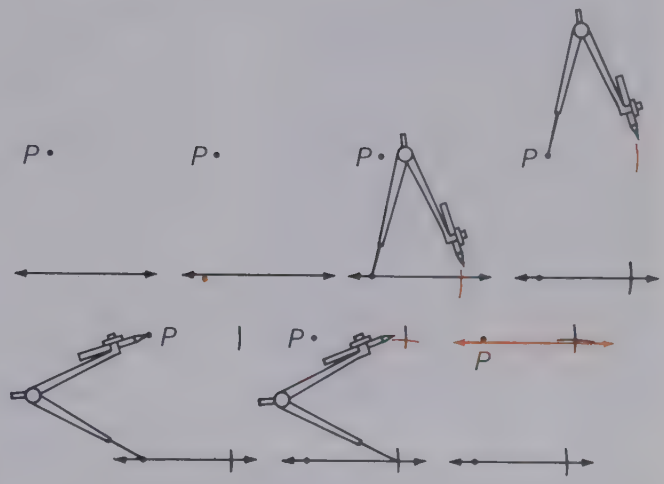
1. Use the flow chart to bisect an angle. Is the chart complete enough to instruct you in how to bisect an angle? If not, can you make improvements?

2. Draw a flow chart for "How to construct a line perpendicular to a given line." If you have forgotten how to do this, you may ask your teacher to do the construction, without comment, on the chalkboard. Then compare your chart with the one below. Which chart do you think is better?


START
DRAW A LINE
MARK 2 POINTS ON A LINE A SUITABLE DISTANCE APART
OPEN COMPASSES TO SUITABLE RADIUS
PLACE POINT OF COMPASSES ON POINT ON LINE
DRAW ARCS ABOVE AND BELOW THE LINE
HAVE YOU MADE TWO SETS OF INTERSECTING ARCS ?
NO
KEEP COMPASSES' RADIUS THE SAME
YES
DRAW A LINE TO CONTAIN THE TWO POINTS OF INTERSECTION OF ARCS
THE LAST LINE IS PERPENDICULAR TO THE FIRST LINE
STOP

3. Construct the flow chart for the following:

   How to construct a line parallel to a given line through a point P not in the line. Use the series of diagrams given, if necessary.

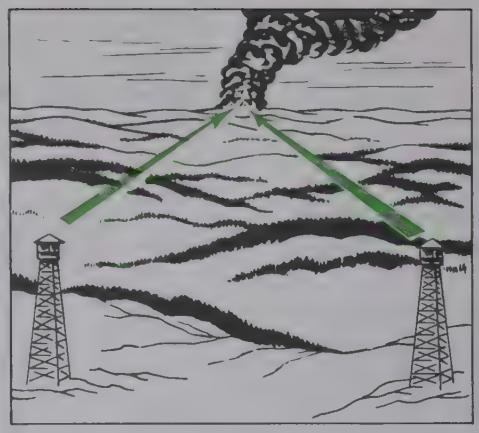


4. Draw flow charts for the following:
   (a) How to construct a perpendicular bisector to a segment.
   (b) How to construct an angle congruent to a given angle.
   (c) How to construct a line perpendicular to a line, through a point P on the line.

5. Draw any triangle. Construct the perpendicular bisector to each side. The three lines should intersect at one point. Use this point as the centre of a circle that circumscribes the triangle (the circle should contain each vertex of the triangle).

6. Draw any triangle. Construct the bisectors of each angle. The three bisectors should intersect at one point. Use this point as the centre of an inscribed circle.

7. Construct an angle whose measure is the sum of the measures of the two angles below.



Fire tower. Ignace, Sioux Lookout District (left). Alidade. Instrument used for determing area of fire (above right). Forestry men sight in on a fire by using triangulation. Can you see how they are able to locate a fire and pinpoint it on a map so they can phone the airbase for a water bomber? You may need to look this up in another book.

Projects

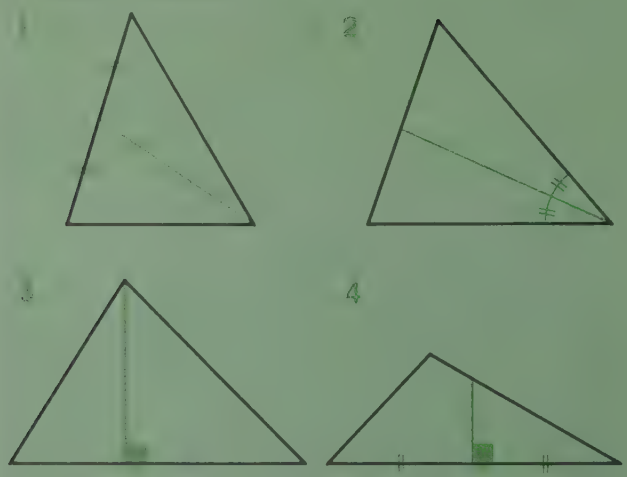
Project 1

Purpose

To study lines of a triangle.

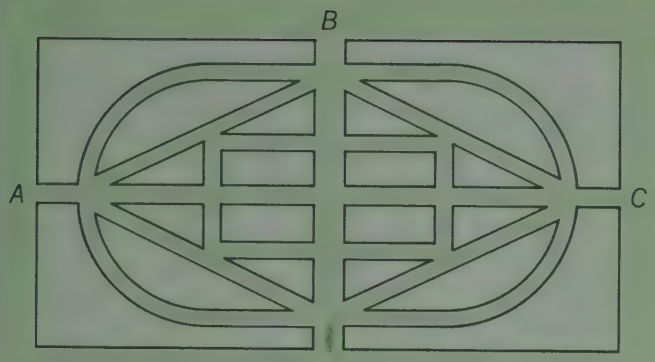
Method

1. Draw 4 triangles.



2. In the first triangle join a vertex and the midpoint of the opposite side. This segment is a *median* of the triangle.
3. Repeat number 2 for the other two vertices.
4. Do the three medians intersect in one point?
5. A median is divided into two segments by this point of intersection. Measure the two segments. What is the ratio of the measure of the shorter segment to that of the longer segment?
6. Repeat number 5 for the other two medians. Is there a constant ratio? Can you write a rule?
7. In the second triangle bisect each angle of the triangle.
8. Do the segments intersect in one point?
9. With the point of intersection as a centre can you use compasses to draw a circle that just touches each side of the triangle? Write as a rule what you have learned.
10. In triangle 3, draw the perpendicular to a side through the vertex opposite the side. This is an *altitude* of the triangle.
11. Repeat number 10 for the other sides.
12. Do the three altitudes intersect at one point?
13. In triangle 4, construct the perpendicular bisector of each side.
14. Do the three lines intersect in one point?
15. Can you draw a circle with this point as centre that contains the three vertices? Write as a rule what you have learned.

Project 2



1. Copy the maze shown above. This is a garden maze. Enter at the gate indicated. Walk *all* paths *exactly* once and come out at *B*.
2. Try to walk all the paths exactly once and come out at *A* or *C*.

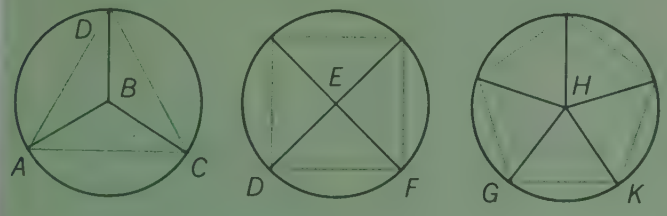
Project 3

Purpose

To draw regular polygons: equilateral triangle, square, pentagon, hexagon and octagon.

Background

We will use sketches of the first three to help us. Sketch 3 circles with a regular triangle in the first, square in the second and a regular pentagon in the third.



Sketch in the radius to each vertex. The triangles so formed in a polygon have the same measures. That is, for example,

$$m\angle ABC = m\angle ABD = m\angle DBC$$

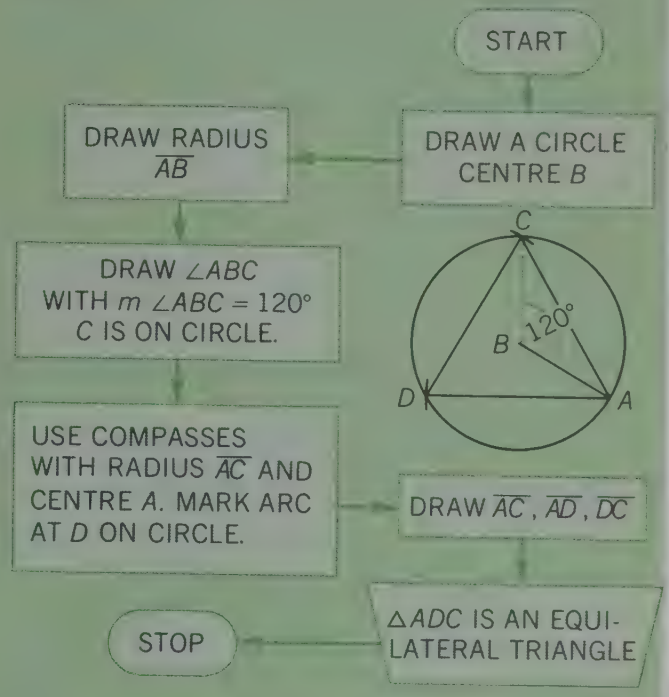
and $m(\overline{AD}) = m(\overline{AC}) = m(\overline{DC})$

and $m(\overline{AB}) = m(\overline{BD}) = m(\overline{BC})$. Why?

To find $m\angle ABC$ we divide 360° by 3; Why?
to find $m\angle DEF$ we divide 360° by 4;
to find $m\angle GHK$ we divide 360° by 5.

The following steps are outlined for the construction of a regular triangle only.

Method



1. Draw the other regular polygons listed.
2. Can you devise an easy way to construct a regular hexagon using only compasses and straightedge? *Hint!* In the hexagon drawn in part 6, compare the measure of the radius with the measure of the side of the hexagon.

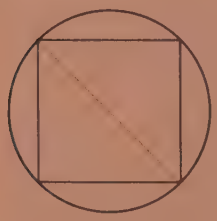
Project 4

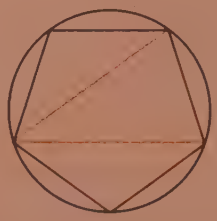
Purpose

To find the measure of the interior angle of a regular polygon. See the figure below for the meaning of interior angle of a polygon.

Method

1. Using the method of Project 3, page 203, draw the following regular polygons: equilateral triangle, square, pentagon, hexagon and octagon.
2. Measure the interior angle of each polygon.
3. From one vertex of each polygon (except the triangle) draw in the diagonals as shown.





4. Complete the table for the first five polygons and apply the pattern you found to the last three polygons.

Project 5

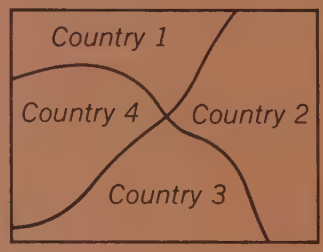
Purpose

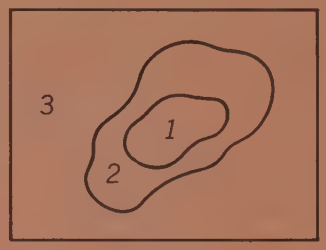
To find the minimum number of colors necessary to color a map in such a manner that no two adjacent areas with a common border have the same color.

Method

1. Copy the following figures in set A. Each is a map.

Set A



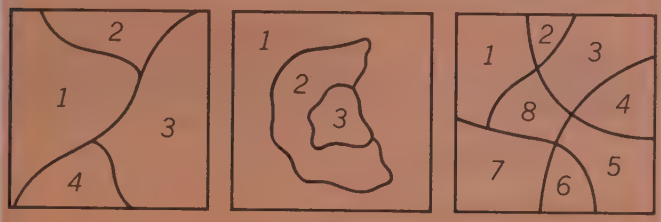


2. Color the regions with a minimum of colors so that no two areas with a common border have the same color. It is permissible for regions of the same color to meet at a point so long as there is no border line in common.

REGULAR POLYGON	NUMBER OF SIDES	ACTUAL MEASURE OF INTERIOR ANGLE	NUMBER OF TRIANGLES	SUM OF MEASURES OF ALL INTERIOR ANGLES	MEASURE OF INTERIOR ANGLE
EQUILATERAL TRIANGLE	3	60	1	1×180	$\frac{1 \times 180}{3} = 60$
SQUARE	4	90	2	2×180	$\frac{2 \times 180}{4} = 90$
PENTAGON					
HEXAGON					
OCTAGON					
DECAGON	10				
DODECAGON	12				
N-GON	*N*				

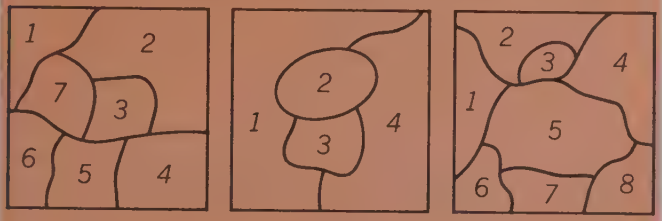
3. What is the minimum number of colors needed for the maps in set *A*?

4. Copy the figures in set *B*. Repeat steps 2 and 3.



Set B

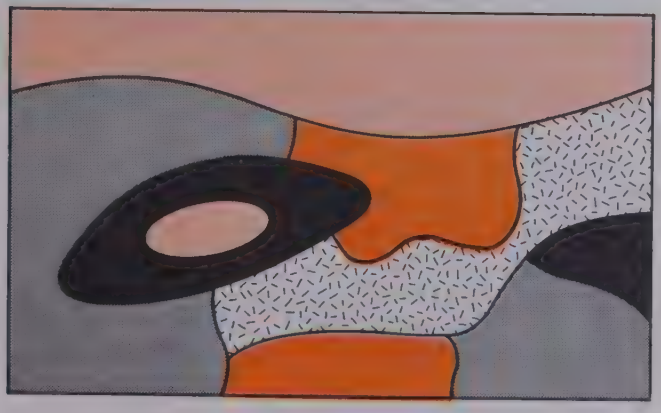
5. Copy the maps in set *C*. Repeat steps 2 and 3.



Set C

6. Can you design a map that requires 5 colors?

7. As yet no design has been found that requires more than 4 colors. It has been proved that no more than 5 colors are needed to color any map (proved by theoretical mathematics). While it seems from actual trials that no more than 4 colors are required, there is no proof of this. Not all problems in mathematics are solved yet!

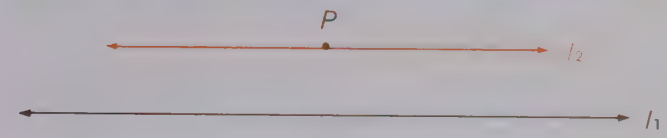


This map is colored with 5 colors. Can you do it with fewer?

9·5 Parallel Lines

It was mentioned in the introduction to this chapter that there are various geometries. It is in the treatment of parallelism that the three geometries diverge. The so-called Euclidean Geometry, named after Euclid, is based on the axiom:

> In a plane containing a line l_1, and a point P not in line l_1, there exists exactly one line that contains P and is parallel to l_1.



Line l_2 is the only line through P that is parallel to l_1.

Euclidean geometry is the geometry most often studied and referred to in high schools. The plane geometry that Euclid organized so well in his books *Elements* is sometimes referred to as *flat* geometry since it applies to the flat regions.

We shall use Euclid's axiom of parallelism. First we shall review the definition of parallel lines.

> Two lines, l_1 and l_2, are parallel if they are coplanar and if they do not intersect.

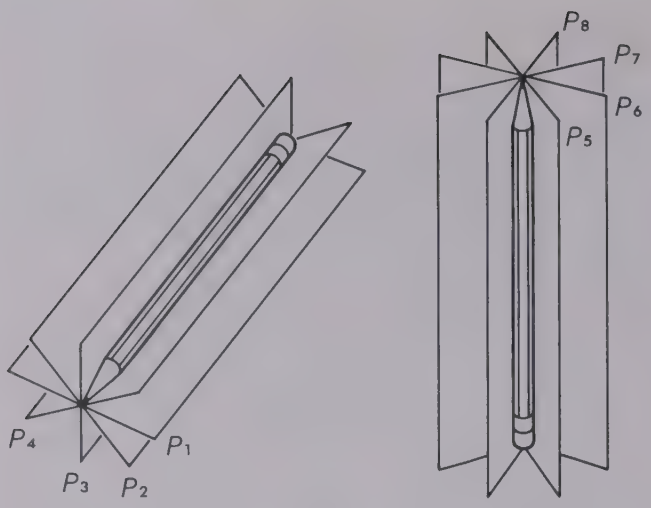
The symbol for parallel is $\parallel$.

Example:
$l_1 \parallel l_2$ is read "Line l_1 is parallel to line l_2."

Since rays and line segments are subsets of lines we state this definition:

> Two line segments or rays are parallel if the lines in which they are subsets are parallel.

We might ask, why do we need the words "they are coplanar" in the definition of parallel lines? One way to realize the significance of these words is through the use of a model.



(a) (b)

Figure 9-5

Hold a pencil horizontally with the sharpened end towards you. Envision some of the planes containing the line represented by the pencil. See Figure 9–5(a). Now hold another pencil vertically to the right of the first pencil. Envision the planes containing the second line represented by the second pencil. See Figure 9–5(b). Is there a plane containing the first line that coincides with a plane containing the second line? Is there a plane that contains both lines? Here there are two lines which do not intersect yet they are not parallel. Two lines which do not intersect and are not parallel are called *skew* lines. See Figure 9–6.

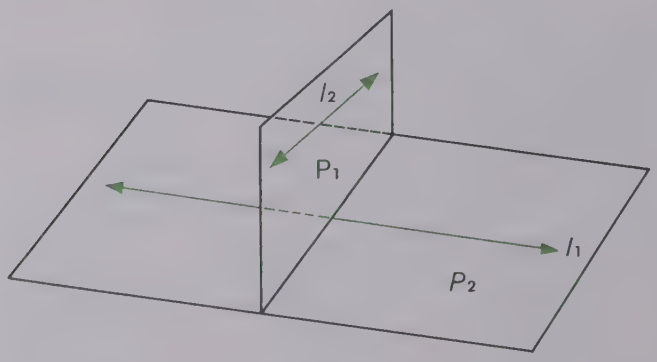
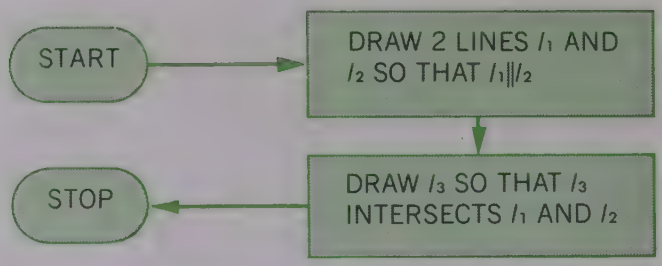


Figure 9-6

Let us now review briefly the properties associated with parallel lines.



Line l_3 is a transversal.

Label the angles as in Figure 9–7.

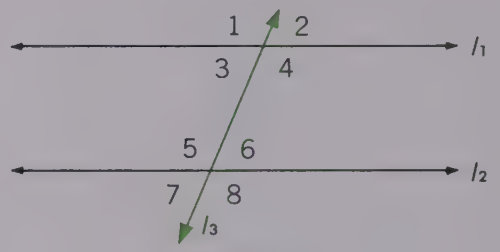


Figure 9-7

Use a protractor to measure the following angles.

$\begin{cases} m\angle 3 = \blacksquare \\ m\angle 6 = \blacksquare \end{cases}$ $\begin{cases} m\angle 4 = \blacksquare \\ m\angle 5 = \blacksquare \end{cases}$

Angles 3 and 6 are *alternate interior angles.* Compare $m\angle 3$ and $m\angle 6$.

Angles 4 and 5 are alternate interior angles. Compare $m\angle 4$ and $m\angle 5$.

Rule 1

> If a transversal intersects two parallel lines then the measures of the alternate interior angles so formed are equal.

Measure the following angles.

$\begin{cases} m\angle 1 = \blacksquare \\ m\angle 5 = \blacksquare \end{cases}$ $\begin{cases} m\angle 2 = \blacksquare \\ m\angle 6 = \blacksquare \end{cases}$

$\begin{cases} m\angle 3 = \blacksquare \\ m\angle 7 = \blacksquare \end{cases}$ $\begin{cases} m\angle 4 = \blacksquare \\ m\angle 8 = \blacksquare \end{cases}$

Angles 1 and 5 are corresponding angles. Each of the above four pairs is a set of corresponding angles. Compare the measures of the 2 angles in each pair.

Rule 2

> If a transversal intersects two parallel lines, the measures of the corresponding angles so formed are equal.

Record the measures of the following pairs of angles:

$\begin{cases} m\angle 1 = \blacksquare \\ m\angle 4 = \blacksquare \end{cases}$ $\begin{cases} m\angle 2 = \blacksquare \\ m\angle 3 = \blacksquare \end{cases}$

$\begin{cases} m\angle 6 = \blacksquare \\ m\angle 7 = \blacksquare \end{cases}$ $\begin{cases} m\angle 5 = \blacksquare \\ m\angle 8 = \blacksquare \end{cases}$

What do you observe?

Each pair of angles is a pair of *vertically opposite angles.*

Rule 3

> When two lines intersect the measures of the vertically opposite angles are equal.

Follow the reasoning in the next five statements:

$m\angle 1 = m\angle 5$	Why?
$m\angle 4 = m\angle 5$	Why?
$\therefore m\angle 4 = m\angle 1$	Why?
$m\angle 4 = m\angle 8$	Why?
$\therefore m\angle 1 = m\angle 8$	Why?

How many sets of parallel lines can you identify in this picture?

Use your protractor to measure $\angle 1$ and $\angle 8$. Are the measures equal?

Show in a similar manner that $m\angle 2 = m\angle 7$.

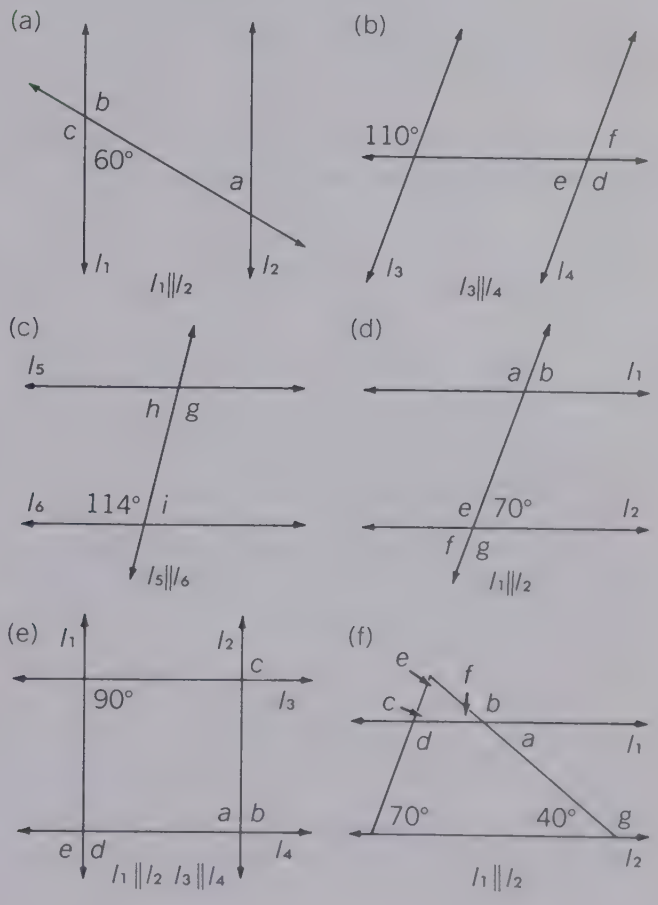
Angles 1 and 8, and angles 2 and 7 are pairs of *alternate exterior angles.*

Rule 4

> If a transversal intersects two parallel lines, the measures of the alternate exterior angles so formed are equal.

EXERCISE 9·5

1. From the data given find the measures of the angles marked by letters.



9·6 Converse

Consider the *if . . . then* statement:

If an Apollo flight is successful, then it will orbit the moon.

The converse of this statement is:

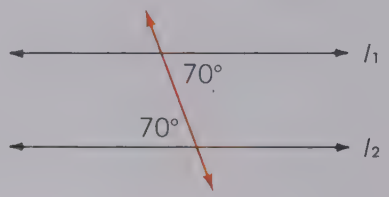
If an Apollo flight orbits the moon, then it will be successful.

Note that just because the original statement is true does not mean the converse is true for, in the example we have, there are a number of ways in which the Apollo flight could fail after orbiting the moon. The converse of some statements is true and of others is false.

Converse of Rule 1

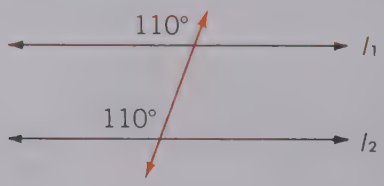
If the measures of the alternate interior angles formed by a transversal intersecting two lines l_1 and l_2 are equal then the two lines l_1 and l_2 are parallel.

Use a protractor to draw accurately the figure below. Do lines l_1 and l_2 appear parallel?



Converse of Rule 2

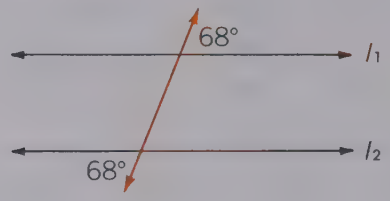
If the measures of two corresponding angles formed by a transversal intersecting two lines l_1 and l_2 are equal then the two lines l_1 and l_2 are parallel.



Converse of Rule 4

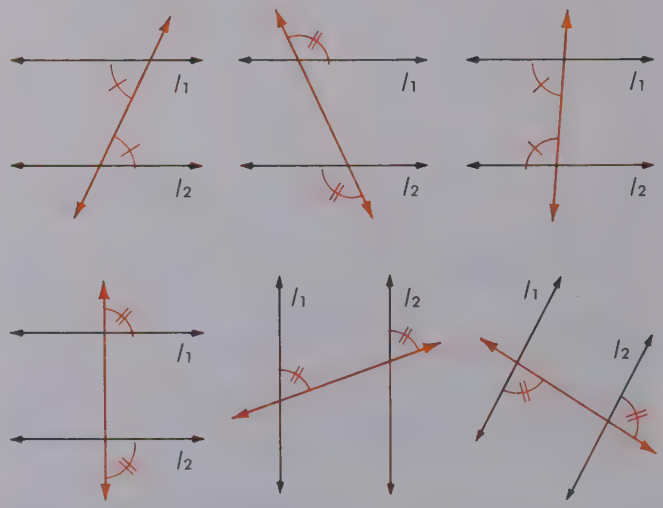
If the measures of two alternate exterior angles formed by a transversal intersecting two lines l_1 and l_2 are equal then the two lines l_1 and l_2 are parallel.

These three converses are true.



EXERCISE 9·6

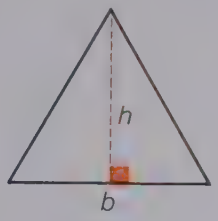
1. Determine whether or not lines l_1 and l_2 are parallel in each illustration. If the lines are parallel, tell which rule applies.



2. Write the converse of each statement. Decide whether or not the statement is true and whether or not the converse is true.
   (a) If I live in Winnipeg, then I live in Canada.
   (b) If Mary goes swimming, then she will have wet hair for the party.
   (c) If Mr. Trudeau is Prime Minister, then Mr. Trudeau is the leader of the Liberal Party.
   (d) If $x = y$ then $y = x$.
   (e) If $m \angle B = 90°$ then $\angle B$ is a right angle.

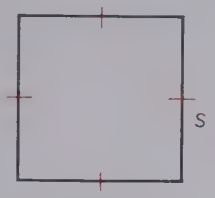
9·7 Parallel Lines at Work

Parallel lines occur frequently in geometric figures, both plane and solid. The common plane figures, some with parallel lines are shown along with the formulas for the measures of the areas of the interior regions.



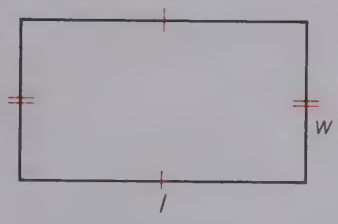
$A = \frac{1}{2}hb$

Triangle



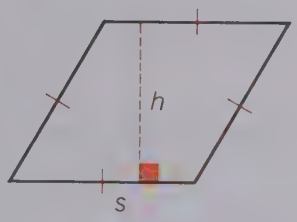
$A = s^2$

Square



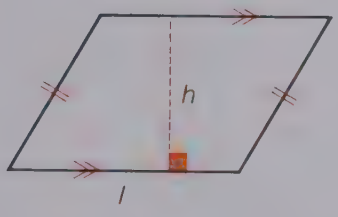
$A = lw$

Rectangle



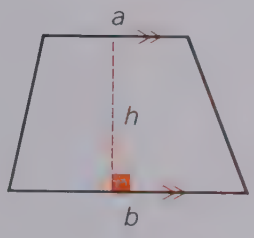
$A = hs$

Rhombus



$A = hl$

Parallelogram



$A = h(?)$

Trapezoid

For Discussion:

1. Is a square a rectangle?

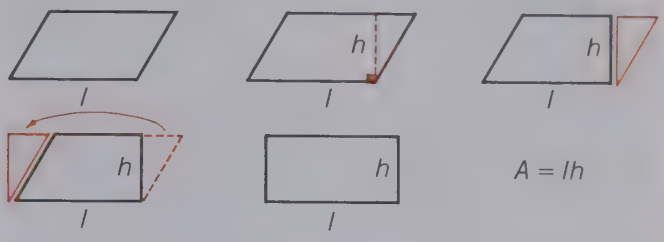
2. What feature does a square have that a rectangle may not have?

3. What features do a square and a rhombus have in common? How do they differ?

4. What features do a rectangle and a parallelogram have in common? How do they differ?

5. What do a trapezoid and a parallelogram have in common? How do they differ?

6. Can you follow the series of diagrams to review the area of the region of a parallelogram?



7. Can you complete the formula for the trapezoid?

$$A = h(?)$$

If your answer is no, do the next question.

8. Study the diagrams below. Notice that Figure $A'B'C'D'$ is a duplicate of $ABCD$

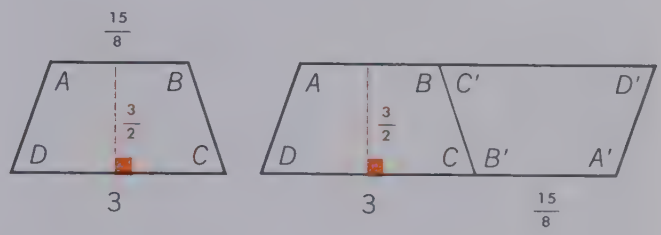


Figure $AD'A'D$ is a parallelogram. The measure of the area of the trapezoidal region $ABCD$ is one-half that of the parallelogram $AD'A'D$. The measure of the area of the parallelogram is $\frac{3}{2}(3 + \frac{15}{8})$. Therefore the measure of the area of the trapezoid is $\frac{1}{2}[\frac{3}{2}(3 + \frac{15}{8})]$. Calculate this value. Did you get $\frac{117}{32}$?

General Formula:

$$A = \frac{1}{2}h(a + b)$$

where a and b are the measures of the two parallel sides and h is the perpendicular distance between them.

Project

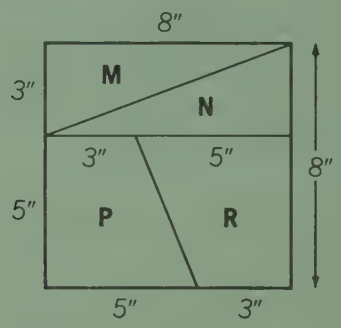
Project

Purpose

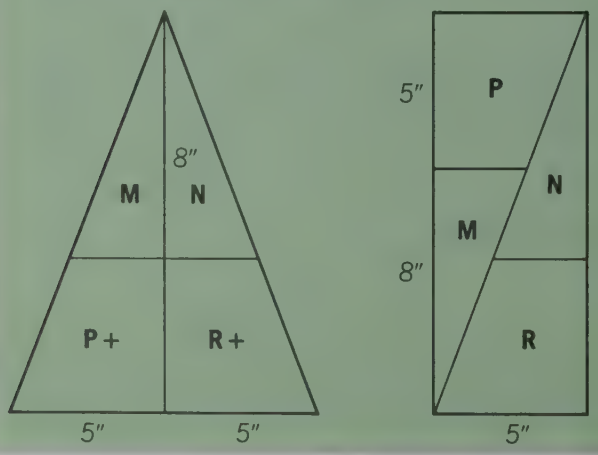
Is it possible for 64 to become 65?

Method

1. Choose an 8″ × 8″ sheet of graph paper with $\frac{1}{10}$ inch ruling.
2. Cut as shown.

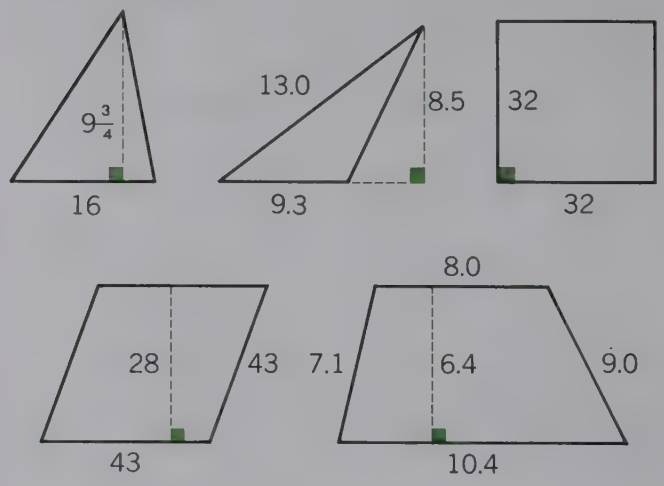


3. Reassemble as in second diagram.
4. What is the area of the square?
5. What is the area of the triangle?
6. Is there any more paper in the triangle than in the square?
7. Do the two answers in 4 and 5 agree?
8. Can you explain why it appears that 64 = 65?
   Hint! Reassemble as in the third diagram. Exercise care!
9. Are things always as they appear?

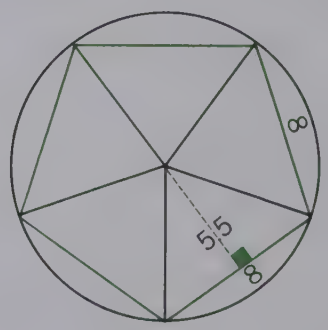


EXERCISE 9·7

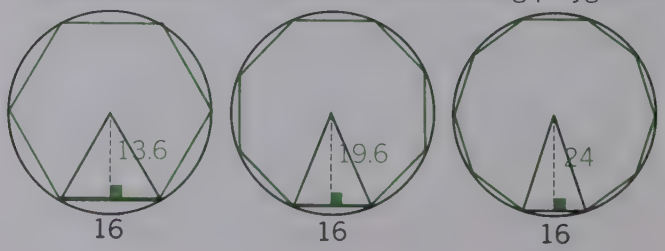
1. Find the measure of the areas of the following regions. Use a slide rule or calculator for quick results.



2. The measure of the area of a regular pentagon is 5 times that of one of the triangles formed in the pentagon. Find the measure of the area of the pentagon from the information given in the figure.



3. Use the principle established in question 2 to find the measure of the area of the following polygons.



9·8 Drawing Parallel Lines

Earlier we constructed a line l_2 parallel to a line l_1 through a point P not on line l_1. Draftsmen use an easier method to draw parallel lines by means of a draftsman's triangle or a parallel ruler. Figure 9-8 shows the method of drawing a line l_2 parallel to line l_1 through a point P not in line l_1 by using a draftsman's triangle. Can you explain how you know l_2 is parallel to l_1?

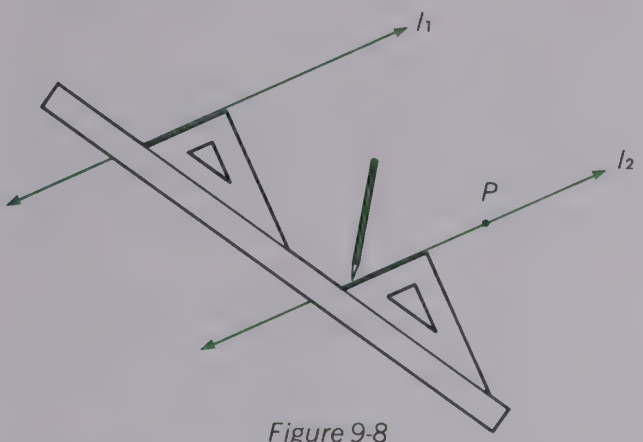
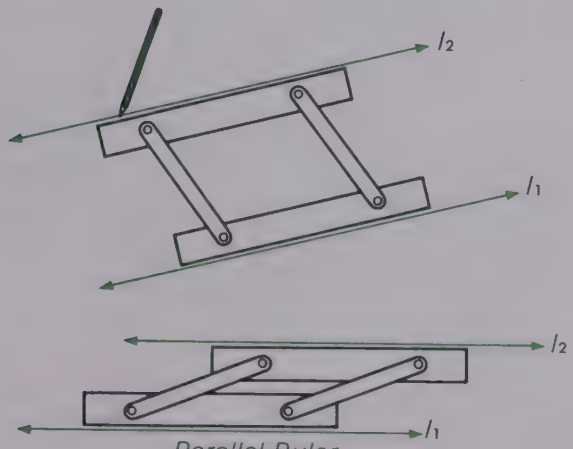


Figure 9-8

Figure 9-9 shows the method of drawing a line l_2 parallel to line l_1 through a point P not in the line l_1 by using a parallel ruler.



Parallel Ruler

Figure 9-9

The lines of your notebook are equally spaced parallel lines.

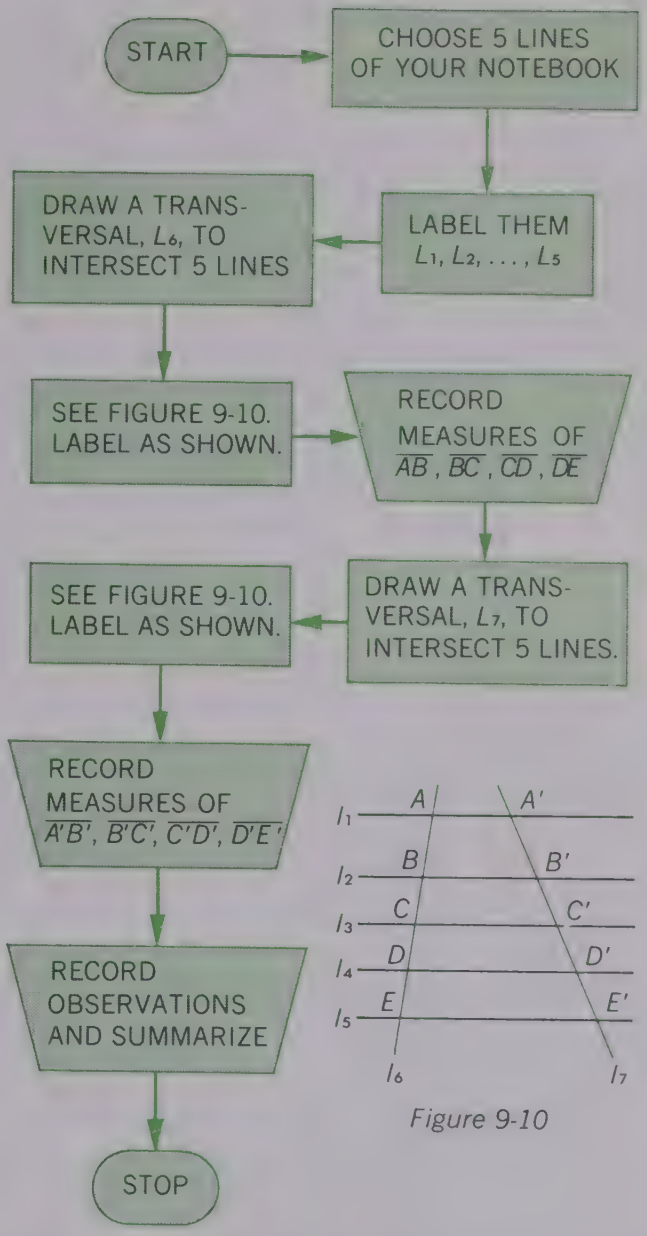


Figure 9-10

This illustration is a verification of the following rule:

Rule 5

> If three or more parallel lines cut off segments with equal measures on one transversal, then they cut off segments with equal measures on any other transversal.

Projects

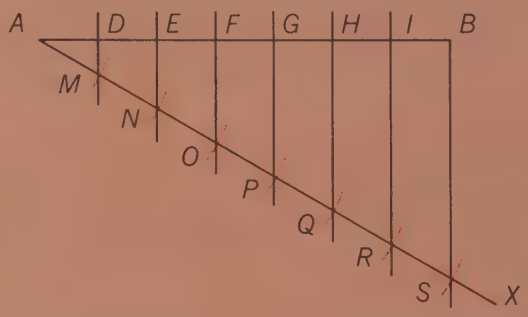
Project 1

Purpose

Divide $\overline{AB}$, with measure of 4 inches, into 7 equal parts.

Method

1. Draw $\overline{AB}$.
2. Draw $\overline{AX}$ at an angle (about 30°) to $\overline{AB}$.



3. With compasses mark off on $\overline{AX}$ seven equal segments; $\overline{AM}$, $\overline{MN}$, $\overline{NO}$, etc.
4. Draw $\overline{SB}$. Using draftsman's triangle or parallel ruler draw through M, N, O etc., lines parallel to $\overline{SB}$ to intersect $\overline{AB}$ in D, E, F, etc., respectively.
5. $\overline{AD} \equiv \overline{DE} \equiv \overline{EF} \equiv \overline{FG} \equiv \overline{GH} \equiv \overline{HI} \equiv \overline{IB}$.
6. Do you see that this Project is based on Rule 5?

Project 2

Purpose

To establish some properties of the special quadrilaterals.

Method

1. Draw carefully one of each of the following: square, rectangle, rhombus, parallelogram and trapezoid.
2. Use a protractor to find the measures of the angles and a ruler to find the measures of the segments.
3. Copy the table below and put a check mark in each column for which the statement is true.
4. Do the first 8, then draw in both diagonals in each figure. Measure. Complete the table.

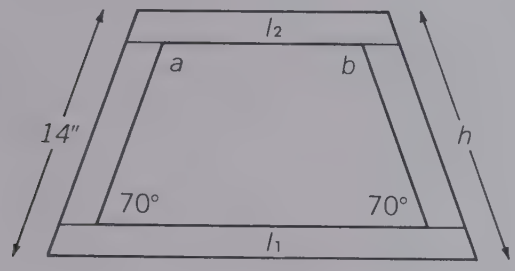
Statement	SQUARE	RECTANGLE	RHOMBUS	PARALLELOGRAM	TRAPEZOID
1. Opposite angles of the quadrilateral are congruent.					
2. Exactly one pair of opposite sides of the quadrilateral are congruent.					
3. Both pairs of opposite sides of the quadrilateral are congruent.					
4. Exactly one pair of opposite sides are parallel.					
5. Both pairs of opposite sides are parallel.					
6. Adjacent sides are congruent.					
7. The quadrilateral has at least one right angle.					
8. The quadrilateral has four right angles.					
9. The diagonals are congruent.					
10. The diagonals bisect each other.					
11. The diagonals intersect each other at right angles.					
12. The diagonals bisect the angles at all the vertices.					
13. The diagonals form alternate angles which are equal with both pairs of sides.					

EXERCISE 9·8

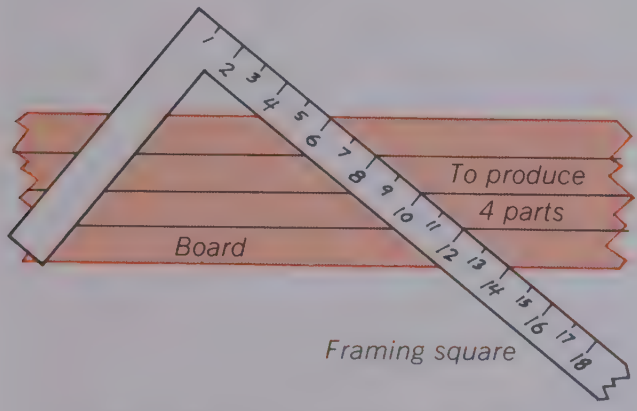
1. Draw a segment $\overline{AB}$ congruent to this segment.

With compasses and straightedge and parallel ruler divide $\overline{AB}$ into 6 equal parts.

2. A tennis racquet press has the dimensions shown below. End l_1 is parallel to end l_2. What is the measure of angle a? b? What is the measure of the side marked h?



3. It is often desirable to divide a board (for ripping) into a certain number of parts. It is a simple matter to lay a square or rule across the board so that the required number of divisions just cover the board. The length of the unit division can be of any length (1inch, 3 inches, etc.) but the divisions must be all the same. If 4 parts are necessary then 4 unit divisions are used.

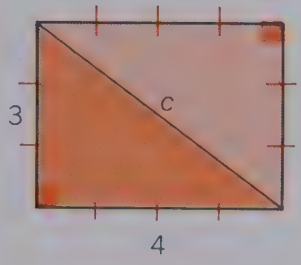


Use this method to divide your page into
(a) 12 equal parts lengthwise.
(b) 4 equal parts lengthwise.
What unit division did you use for each?

9·9 Pythagorean Relationships

The Pythagorean relationship was used in Chapter 2 in the discussion of irrational numbers. It is interesting to look at the geometric representation of the relationship. A book, *The Pythagorean Proposition,* by E. S. Loomis, contains 109 algebraic proofs, 147 geometric proofs and 6 other proofs using advanced mathematics. It is reported that there are more than 350 original proofs for the Pythagorean property. President Garfield, twentieth president of the United States, developed the following proof. Here it is presented using numbers.

A rectangle of measures 3 by 4 is cut along the diagonal and reassembled as shown.



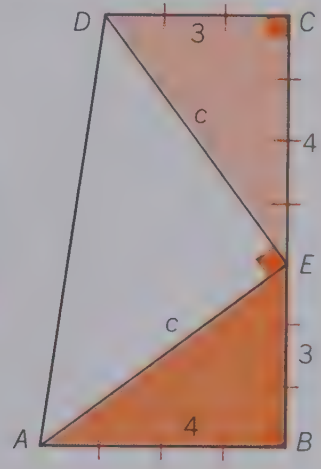


Figure 9-11

The measure of the area of the trapezoid $ABCD$ is equal to the sum of the measures of the areas of the three triangles.

$$m(ABCD) = m\,\triangle ABE + m\,\triangle DCE + m\,\triangle DEA$$
$$\tfrac{1}{2}(4+3)7 = \tfrac{1}{2}(4)(3) + \tfrac{1}{2}(4)(3) + \tfrac{1}{2}(c)(c)$$
$$\tfrac{49}{2} = \tfrac{1}{2}(12 + 12 + c^2)$$
$$49 = 24 + c^2$$
$$c^2 = 25$$
$$c = 5.$$

Now in our original rectangle we have $3^2 + 4^2 = 5^2$. This verifies the statement "The square on the hypotenuse of a right triangle is equal to the sum of the squares on the other two sides."

Projects

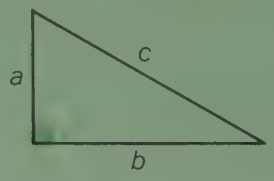
Project 1

Purpose

To verify the Pythagorean Property:

$$a^2 + b^2 = c^2$$

where c, a, b are the hypotenuse and the two sides of a right triangle respectively.



Method

1. On a piece of graph paper, 10 lines to the inch, draw a right triangle with the measures of the two sides 3 and 4 inches.
2. On each side of the triangle draw a square.
3. Determine the measure of the area of the square on each side.
4. Repeat using the measures of the sides:

	a	b	
(ii)	5	12	
(iii)	6	8	
(iv)	8	15	(Draw this one to scale.)

5. Complete this table:

			Area of square on side			Sum of areas of squares on sides a and b
	a	b	a	b	c	
(i)	3	4	9	16		25
(ii)	5	12				
(iii)	6	8				
(iv)	8	15				

6. How do the numbers in the last column compare with those in the second last column? Is your measurement of the area of the square on c accurate?
7. Do the results of (5) verify this statement? "The measure of the area of the square on the hypotenuse is equal to the sum of the measures of the areas of the squares on the other two sides."

Project 2

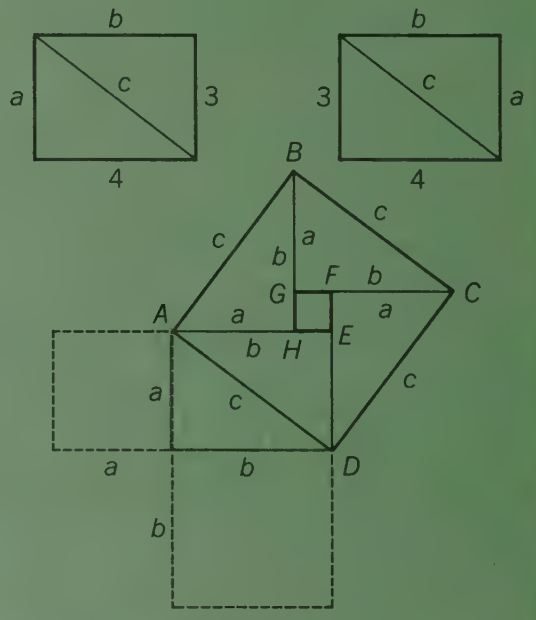
Purpose

To verify the Pythagorean Property.

Method

1. On graph paper draw 2 rectangles with side measures of

	a	b		a	b
(i)	3	4	(ii)	5	12
(iii)	8	15	(iv)	7	24



2. Draw one diagonal in each.
3. Cut out each rectangle to form 2 triangles from each.
4. Reassemble the 4 congruent triangles to form a square $ABCD$ with a square hole in the middle.
5. Total area of square $ABCD$ is equal to the sum of the areas of its parts.

   $m(\overline{AB}) = m(\overline{BC})$.

   Area of square $ABCD$ is c^2.

   Is $\triangle AHB \equiv \triangle BGC$?

   Is $\triangle ABH \equiv \triangle CDF$?

   Is $\triangle ABH \equiv \triangle DEA$?

   What is $m(\overline{AE})$? $m(\overline{AH})$? $m(\overline{HE})$?

   What is the measure of the area of square $HEFG$?

6. Complete the following table.

Trial	1. Area of square $ABCD$ [c^2]	2. Area of $\triangle AHB$	3. Area of 4 triangles congruent to $\triangle AHB$	4. Area of square $GFEH$	5. Sum of columns 3 and 4, [$a^2 + b^2$]
(i)					
(ii)					
(iii)					
(iv)					

7. How does each number in column 1 compare with its corresponding number in column 5?

8. Do the results in this table tend to verify that $a^2 + b^2 = c^2$?

Project 3

Purpose

To verify Pythagorean Triples.

Background

A set of 3 numbers which satisfies the Pythagorean Property is called a Pythagorean Triple. The 3 numbers 3, 4, 5 are a Pythagorean Triple since $3^2 + 4^2 = 5^2$.

Method

1. The following table lists some Pythagorean Triples. Some of the triples are not Pythagorean. Complete the table to determine which are Pythagorean Triples.

a	b	c	$a^2 + b^2$	c^2	Pythagorean Triple
3	4	5	25	25	yes
3	6	8	45	64	no
5	12	13			
8	15	17			
7	24	25			
9	21	23			
20	21	29			
16	30	34			
12	35	37			
27	36	45			
31	35	48			
11	48	50			
14	48	50			
119	120	169			

The converse of the Pythagorean Property states

> If the square of the measure of the longest side of a triangle is equal to the sum of the squares of the measures of the other two sides, then the triangle is a right triangle.

This converse is used to determine if a triangle is a right triangle.

Example 1

The measures of the sides of a triangle are 4, 6 and 8. Is this a right triangle?

Solution

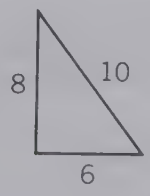
Test to see if

$$8^2 \stackrel{?}{=} 4^2 + 6^2$$
$$64 \stackrel{?}{} 16 + 36$$
$$64 \neq 52.$$

$\therefore$ this triangle is not a right triangle.

Example 2

The measures of a triangle are as shown. Is this a right triangle?



Solution

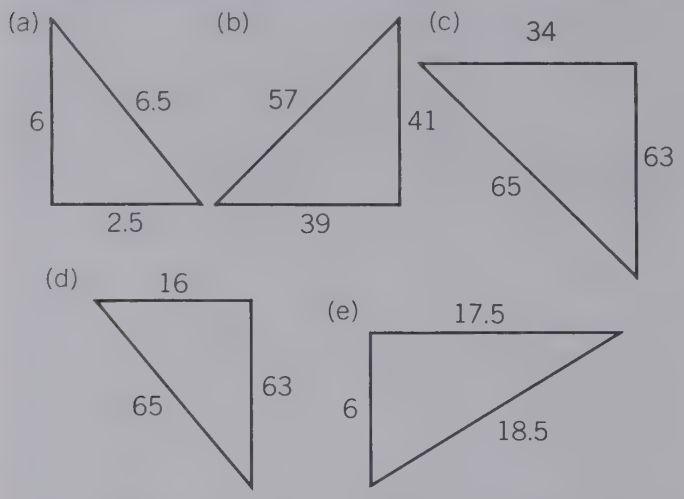
Test to see if

$$10^2 \stackrel{?}{=} 8^2 + 6^2$$
$$100 \stackrel{?}{} 64 + 36$$
$$100 = 100.$$

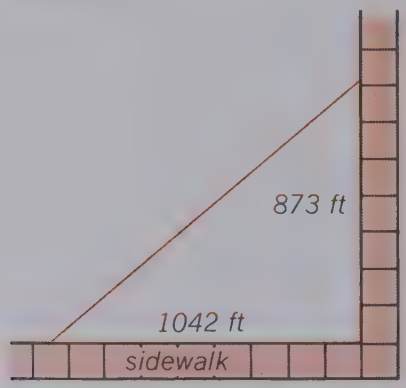
$\therefore$ this is a right triangle.

EXERCISE 9·9

1. Which of the following triangles are right triangles?



2. In walking to school, John has the option of walking on the sidewalk as shown or walking diagonally across the block.



   (a) How much shorter is it to walk across rather than around?

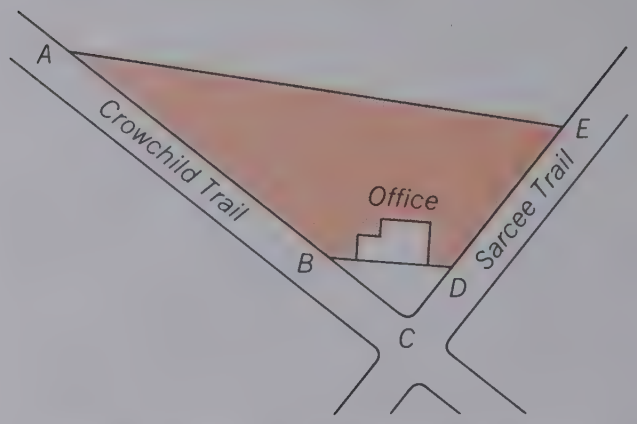
   (b) If John walks 3 miles per hour on the sidewalk but 2.6 miles per hour on the diagonal does he save any time by walking on the diagonal?

   (c) How long does it take him to walk on the sidewalk? on the diagonal?

3. The rafters of a house are 20 feet long and the ridge of the roof is 13 feet above the plate the rafters sit on. How wide is the house?

4. An automobile club locates on the corner of Crowchild Trail and Sarcee Trail. The dimensions in feet of the corner lot are:

   $\overline{AB} = 200$ $\overline{BC} = 60$ $\overline{CD} = 40$
   $\overline{DE} = 110$ $m \angle BCD = 90°$

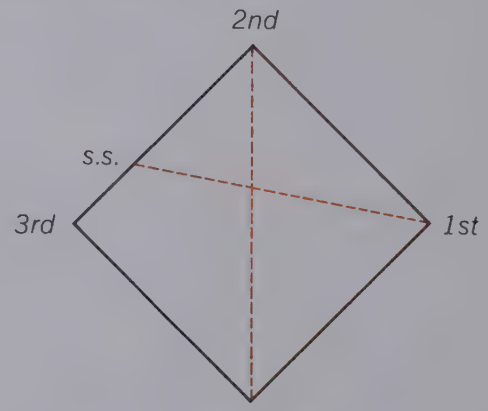


   Find the area of the shaded parking lot if the office building occupies 1400 square feet.

5. A baseball diamond is a square with measure 90 feet on each side.

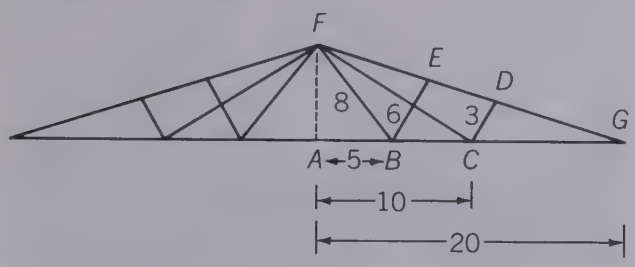
   (a) How far is the throw from the catcher at home plate to second base?

   (b) The shortstop stands on the baseline between second and third and 30 feet from third base. He throws the ball to first base. What is the length of the throw?



6. Design the floor plan for a house. Obtain the price of broadloom from a catalogue or sale advertisement in the newspaper. How much would it cost to cover the floors of the house you planned with broadloom? Would you put the same broadloom throughout the house?

7. A bridge has the truss structure as shown. From the given data find the required measures. Measures are in feet.



Given: $m(\overline{FB}) = 8$, $m(\overline{AB}) = 5$, $m(\overline{BE}) = 6$,
$m(\overline{AC}) = 10$, $m(\overline{CD}) = 3$, $m(\overline{AG}) = 20$

Find

(a) the measures of $\overline{FA}$, $\overline{FC}$, $\overline{FE}$, $\overline{FD}$, and $\overline{FG}$,

(b) the total linear measure of trusses in the portion of the bridge shown. Include the plate the bridge sits on.

The next three problems are actual problems (modified in some details) that an employee of a lumber company was required to solve. He had difficulty and had asked a student for help. If you were the student could you help him? Solve the three problems.

8. A Board Measure (B.M.) of lumber is defined as a piece of material 1 inch thick by 1 foot wide by 1 foot long or its equivalent.

B.M. = thickness (in inches) × width (in feet) × length (in feet)

How many Board Measures (B.M.) of lumber are in each order?

(a) 23 pieces, $1\frac{3}{4}'' \times 7\frac{3}{4}'' \times 20'$

(b) 46 pieces, $2\frac{1}{2}'' \times 5'' \times 12'$

(c) 155 pieces, $3\frac{3}{4}'' \times 9\frac{3}{4}'' \times 14'$

(d) 155 pieces, $4'' \times 10'' \times 14'$

(e) 214 pieces, $\frac{3}{4}'' \times 3\frac{3}{4}'' \times 16'$

(f) 214 pieces, $1'' \times 4'' \times 16'$

(g) Lumber is sold at *full* dimensions. Its actual size is called *dressed* dimensions. That is, a board with actual dimensions $1\frac{3}{4}'' \times 7\frac{3}{4}''$ is called and sold as a $2'' \times 8''$ or a 2×8. If it is 12 feet long, its B.M. is calculated as

$$2 \times \tfrac{8}{12} \times 12 = 16.$$

Part (d) above is the *full* measure of the material in (c). What percent less is the B.M. of (c) than of (d)?

(h) What percent less is the B.M. of (e) than of (f)?

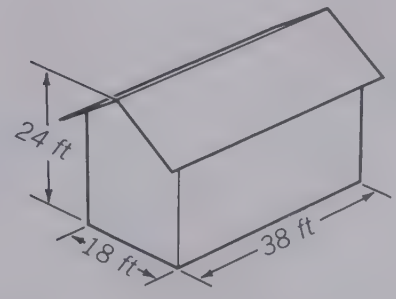
(i) Does one lose more in buying small stock (1×4, 1×6) or by buying large stock (1×8, 2×10)?

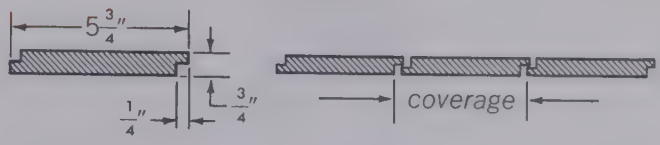
(j) Full dimension is the whole inch dimension. That is, $1\frac{1}{2}''$ is considered $2''$. Find the full dimension B.M. of (a) and (b) above.

9. How many Board Measures 3×6 decking are required to cover a surface of 1700 square feet if the actual covering width of the decking is 5″ due to loss by *dressing* and tongue and groove allowance?

10. A customer is putting sheeting on the frame of a house (general shape is shown below). The roof is gable with 24″ eaves on sides and 18″ overhang on the ends. The size of the house is 18 feet by 38 feet by 24 feet. Roof pitch is $\frac{4}{12}$.

(a) How many Board Measures of 1 by 6 must the customer receive?





(b) The boards are 1×6 spruce shiplap. The cross section of a shiplap board is shown above. While the board is sold as a 6-inch board the coverage is not 6 inches. Find the percentage lost due to manufacturing (dressing) and lap of the boards. Find how many B.M. will actually have to be ordered.

9·10 Construction of Models

Models in geometry can be constructed from many different media. While different media will be mentioned from time to time detailed reference will be made to

(a) pipe cleaners and drinking straws, and

(b) Bristol board (good quality cardboard) and heavy paper.

Much success has been experienced with styrofoam, wood dowels, plywood, plastics, plexiglass, floor tiles and many other materials.

(a) Use pipe cleaners and drinking straws. These are used to make skeletons of models. To join two straws, a pipe cleaner is cut in half, one half bent to the required angle and inserted into the straws (Figure 9-12(a)). The ends are bent as shown, to hold better. To join more than two straws bend the pipe cleaner as shown, then insert into the straws (Figure 9-12(b), (c)). The straws are cut into two congruent parts for most models.

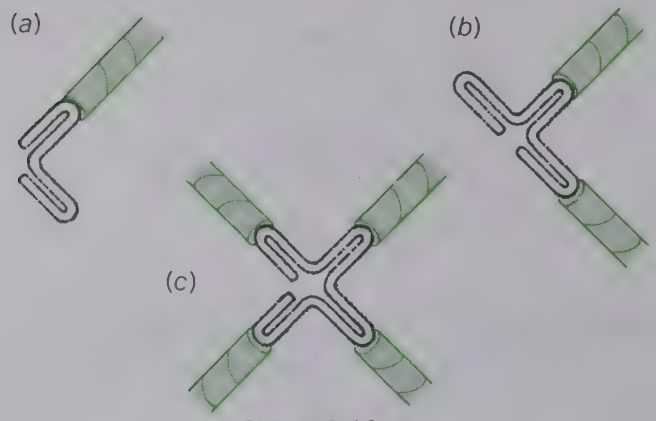
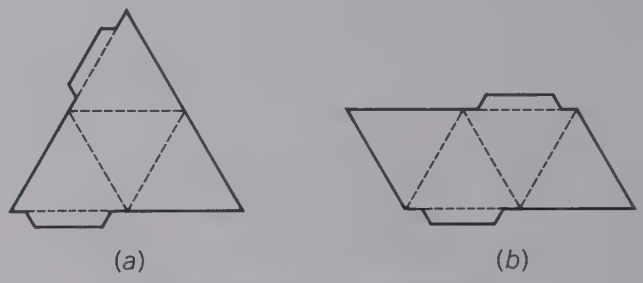


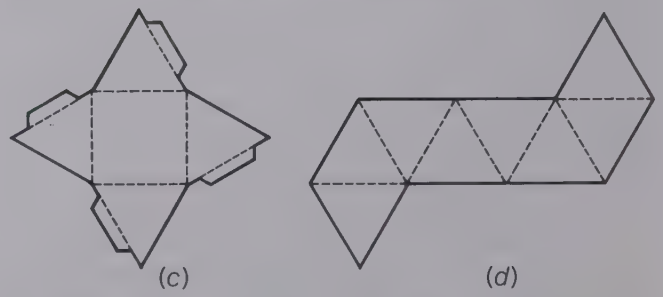
Figure 9-12

(b) To construct a model from cardboard, lay out the plan (called *net*) on the cardboard, cut out the figure, score with a knife or compasses point along the fold lines, fold and tape or glue. If you use glue, tabs must be provided on the nets. If you wish to use tape, tabs need not be provided.

The Alexander Graham Bell Museum at Braddock, Nova Scotia. The design of the museum and the monument is based on the four-sided pyramid, the tetrahedron.



Two nets for a tetrahedron.



A square-based pyramid.

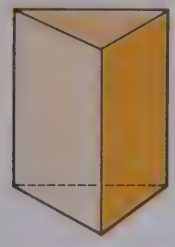
A regular octahedron (tabs not included).

Figure 9-13

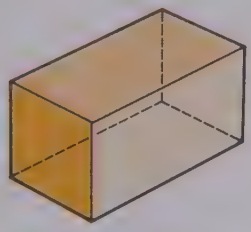
Polyhedra

Many boxes or cartons are models of geometric figures. Most chocolate boxes are models of rectangular *prisms*. Other rectangular prisms are shoe boxes and cornflake boxes. Hat boxes often are hexagonal or octagonal prisms. Some other common prisms are triangular prisms, parallelepipeds and various polygonal prisms.

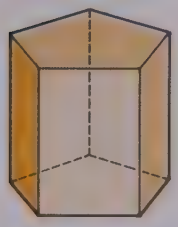
> Prisms have polygonal ends with rectangular sides perpendicular to the ends.



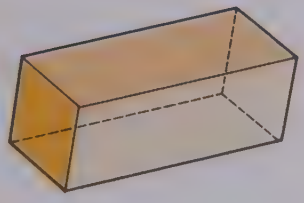
Triangular Prism



Rectangular Prism



Pentagonal Prism

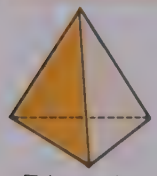


Parallellepiped

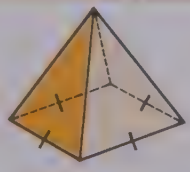
Figure 9-14

When the term *pyramid* is used, most people immediately think of the pyramids in Egypt. These pyramids have square bases with four triangular faces. Other pyramids have polygonal bases. All pyramids have triangular faces.

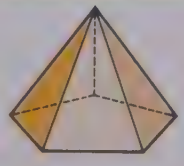
> A tetrahedron (singular for tetrahedra) is a pyramid with a triangular base.



Triangular Pyramid



Square-based Pyramid



Pentagonal Pyramid

Figure 9-15

Prisms and pyramids are members of the set of *polyhedra*. The word polyhedron (singular) comes from Greek words meaning many faces. A polyhedron is any closed solid formed by joining plane polygonal regions edge to edge. Any polyhedron in which all the faces are congruent regular polygons, is called a regular polyhedron. There are only five regular polyhedra. They are:

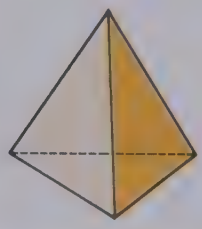
The Regular Tetrahedron (4 faces)
All faces are congruent equilateral triangles.

The Regular Hexahedron (6 faces)
This is the cube. All faces are congruent squares.

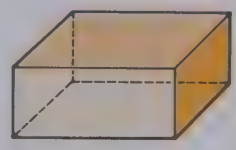
The Regular Octahedron (8 faces)
All faces are congruent equilateral triangles. Four edges meet at each vertex.

The Regular Dodecahedron (12 faces)
All faces are congruent regular pentagons. Three edges meet at each vertex.

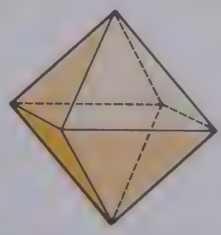
The Regular Icosahedron (20 faces)
All faces are congruent equilateral triangles. Five edges meet at each vertex.



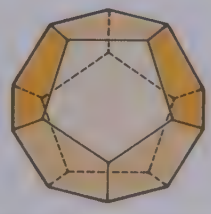
Regular Tetrahedron



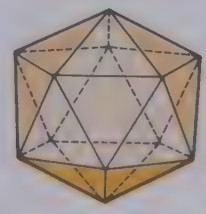
Regular Octahedron



Regular Hexahedron

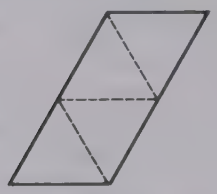


Regular Dodecahedron

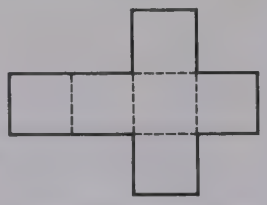


Regular Icosahedron

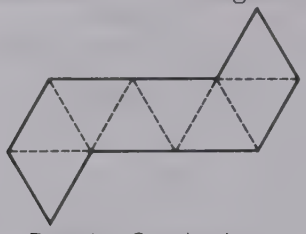
Nets for the five regular polyhedra are shown.



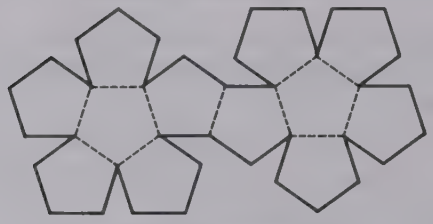
Regular Tetrahedron



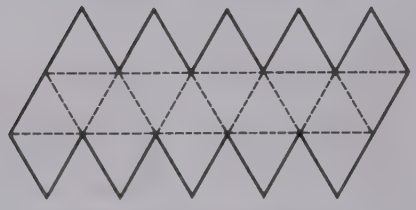
Regular Hexahedron



Regular Octahedron



Regular Dodecahedron



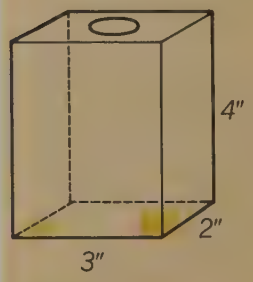
Regular Icosahedron

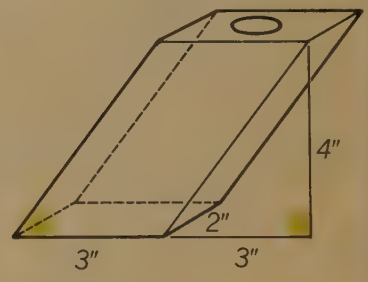
Projects

Project 1

Purpose

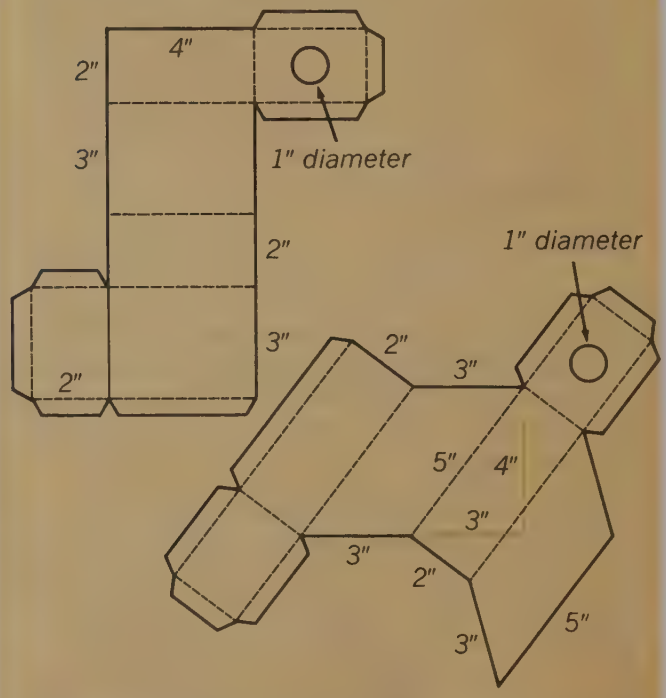
To compare the volumes of two polyhedra shown.





Method

1. Construct the two models by drawing the nets shown, cutting out, folding and gluing. Use Bristol board or heavy cardboard. The colored lines in the net of the parallelepiped are shown only to help you get the proper angle. (Tape the edges so the bird seed will not run out.)



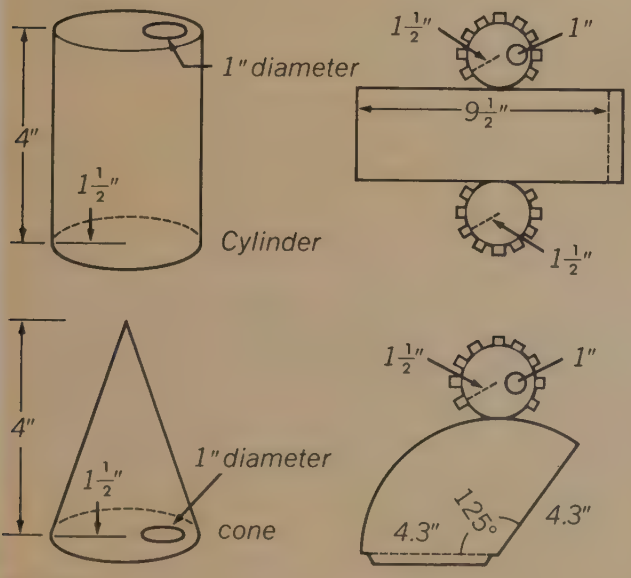
2. Fill the rectangular prism with bird seed or other suitable substance. Then pour this bird seed into the parallelepiped. What is the relation between the two volumes?

3. What is the formula for the volume of a rectangular prism?
4. Is the vertical height of the parallelepiped the same as the height of the rectangular prism?
5. Is the area of the base of the parallelepiped the same as the area of the base of the rectangular prism?
6. What is the formula for the volume of the parallelepiped?
7. Do you think another parallelepiped with the same base area and same vertical height but with more slant would have the same volume as these two polyhedra?
8. Can you construct another parallelepiped similar to the one above with the same volume as the first?

Project 2

Purpose

To compare the volumes of the models shown (a cylindrical prism and a cone).



Method

1. Construct the models from the nets shown.
2. Fill the cone with bird seed and pour it into the cylindrical prism. Repeat until cylindrical prism is full.
3. How many cones full of bird seed were needed to fill the cylindrical prism?
4. The measure of the volume of the cone is what fraction of the measure of the volume of the cylindrical prism?
5. What is the measure of the area of the base of the cylindrical prism? of the cone?
6. What is the measure of the height of the cylindrical prism? of the cone?
7. What is the formula for the volume of a cylindrical prism? for a cone?

Project 3

Purpose

To compare the volumes of a rectangular prism and a square base pyramid.

Method

Repeat steps 1 through 7 of Project 2 replacing pyramid for cone and rectangular prism for cylindrical prism.

Forestry men must be able to approximate the volume, in board measure, of a tree. Suggest several methods of calculating the volume of a tree.

Project 4

Purpose

To get acquainted with polyhedra.
To prepare for Project 5.

Method

1. Construct the skeletons of the following polyhedra from pipe cleaners and straws:
   tetrahedron — triangular prism
   cube — pentagonal prism
   octahedron — dodecahedron
   icosahedron.
   Cut each straw into 2 congruent parts for all models except the dodecahedron and the icosahedron. For the dodecahedron and the icosahedron cut the straws into 4 congruent parts. For diagrams of each model refer to earlier parts of this section.

2. Construct the polyhedra in part 1 from cardboard. Draw accurately the net patterns shown earlier. An appropriate measure for each edge of the models is 4 (inches). Do you remember how to draw a regular pentagon (for the dodecahedron)? If not, refer to Project 3, page 203.

3. Name the shapes in the Canadian Pavilion at Expo '70 in Osaka, Japan.

Canadian Pavilion, Expo 70, Osaka, Japan.

Project 5

Purpose

To discover the relation between the number of edges, the number of faces and the number of vertices of a polyhedron.

Method

1. Use the models built in Project 4.

2. Complete the table.

TYPE OF POLYHEDRA	NUMBER OF FACES *(F)*	VERTICES *(V)*	EDGES *(E)*
triangular prism			
cube	6	8	12
pentagonal prism			
octahedron			
dodecahedron			
icosahedron			

3. Can you spot the relation between *F*, *V*, and *E*.

4. This relation is known as Euler's Relation.

Project 6

Purpose

To determine the minimum number of colors necessary to color a polyhedron if adjacent faces are not to have the same color.

Method

1. Use the polyhedra from the previous projects.
2. How many colors are required to color a tetrahedron? a cube? a triangular prism? a pentagonal prism? an octahedron? a dodecahedron? an icosahedron?
3. What apparently is the minimum number of colors required?
4. Use a styrofoam ball (sphere) or a plain slate globe. Can you design a map on the surface of a sphere that requires 3 colors? 4 colors? 5 colors?
5. Refer to Project 5 in Section 9·4.

Project 7

Purpose

To become acquainted with and understand the concept of the Great Circle route used in navigation.

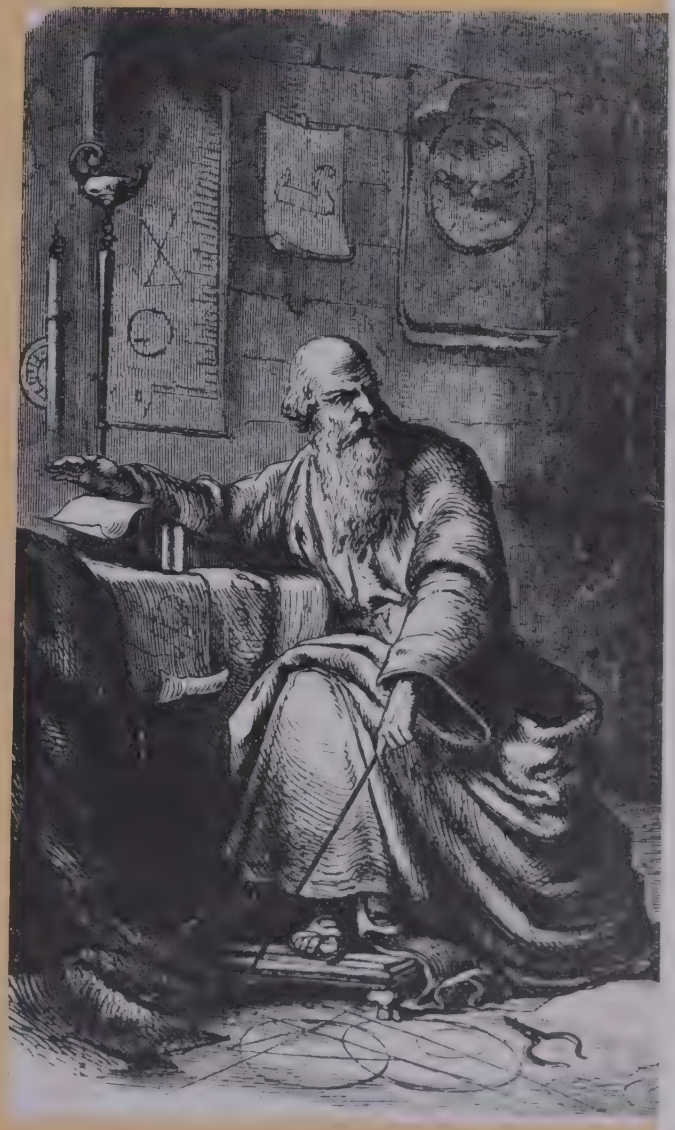
Method

1. Obtain a globe.
2. Edmonton, Alberta and Moscow, U.S.S.R. are on approximately the same latitude. The flight distance along the line of latitude from Edmonton to Moscow is approximately 6300 miles. Use a string to measure the actual distance on the globe between the two cities along the line of latitude. To calculate the scale of your globe, copy and complete these two statements:

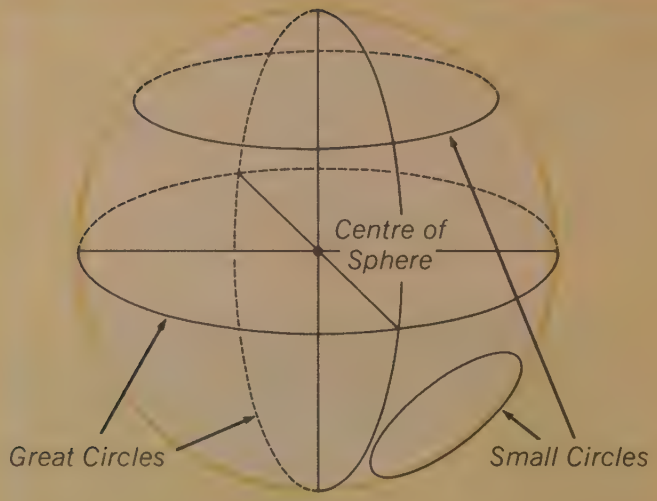
   ■ inches represent 6300 miles.

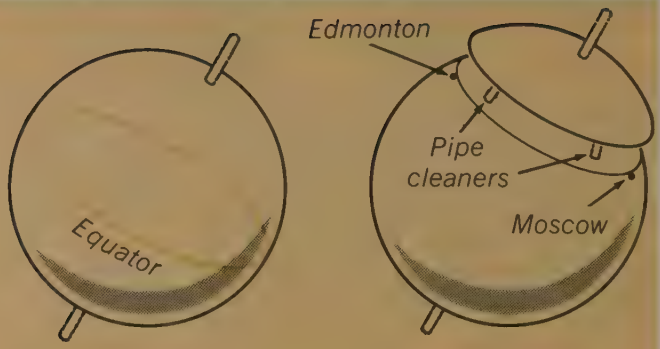
   Therefore, 1 inch represents ■ miles.

   Your globe may have the scale given. Check your answer with the scale given on the globe.
3. Now let the string take the *shortest* distance between these two cities. Is this route along the line of latitude?
4. Name some of the places (countries or cities) the airplane would fly over in flying the *shortest* route between these two cities.
5. What is the measure of the string along the *shortest* route between the two cities?



Archimedes in his study, threatened by a Roman soldier. He is said to have exclaimed "PLEASE DON'T DESTROY MY CIRCLES." Refer to your library to read of the life and death of Archimedes.





6. Use the scale determined in step 2 above to find the *shortest* distance between the two cities.

7. This *shortest* distance is known as the *Great Circle* route. The Great Circle is the intersection of the globe and the plane through the centre of the globe.

8. Use a string on a globe to determine some of the places that an airplane would fly over when on its Great Circle route between the following cities:
   (a) Saigon and Panama.
   (b) Melbourne, Australia and Buenos Aires, Argentina.
   (c) Vancouver, British Columbia and London, England.
   (d) Brazilia, Brazil and Perth, Australia.
   (e) Capetown, South Africa and Auckland, New Zealand.

9. (a) Obtain a styrofoam sphere (6-inch diameter is suitable). Put half a pipe cleaner in the sphere and on the opposite side put in another half pipe cleaner so that these two pipe cleaners represent the North and South Poles. Mark on the line of latitude that represents the equator. Make two dots on the sphere to represent the relative positions of Edmonton and Moscow. Draw in the line of latitude between the two cities. With a sharp knife cut along the line of latitude.

   (b) Repeat part (a) on another sphere but this time cut along the line that represents the *Great Circle*. This cut should go through the centre of the sphere. You should now have two congruent half spheres (called *hemispheres*).

   (c) Which distance is the greater – the distance on the latitude or the distance on the Great Circle – between the two cities?

   (d) The two portions in each case can be held together temporarily by putting short lengths of pipe cleaners into the cutting plane as shown in the diagram above.

Project 8

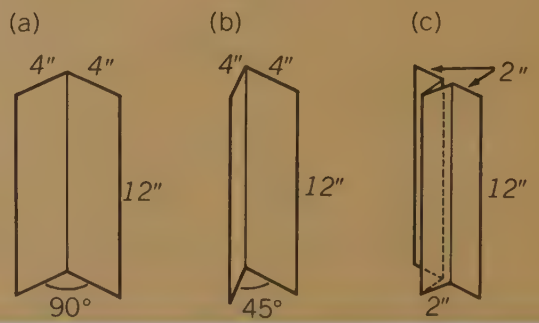
Purpose

To test the strength of various frames.

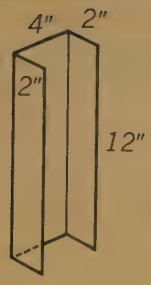
Method

1. Obtain 8 sheets of Bristol board 8″ × 12″. Each sheet must be of the same size and weight.

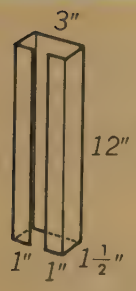
2. Construct the following frames from the sheets.



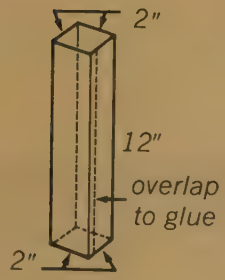
(d)



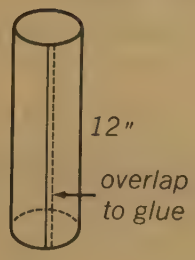
(e)



(f)



(g)



3. Place identical books on each frame until it collapses. Record the maximum number of books that each frame held.

FRAME	NUMBER OF BOOKS HELD
(a)	
(b)	
(c)	
(d)	
(e)	
(f)	
(g)	

4. Each frame had the same amount of material in it. Which frame was the strongest?
5. Can you think of examples in industry or everyday life in which the fact illustrated can be used?

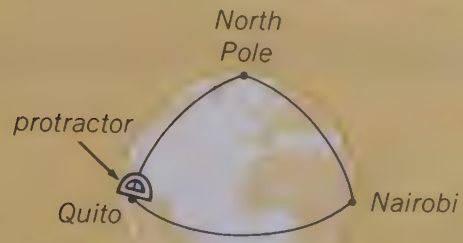
Project 9

Purpose

To study the sum of the angles of a triangle on a sphere.

Method

1. On a globe locate Quito, Ecuador.
2. Draw with chalk the line of longitude through Quito and the North Pole.
3. Locate Nairobi, Kenya.
4. Draw the line of longitude through Nairobi and the North Pole.
5. Quito and Nairobi are on the equator (approximately).
6. We can consider the North Pole, Quito and Nairobi as three vertices of a triangle.
7. What is the measure of the angle at the North Pole? at Quito? at Nairobi? (Measure if necessary.)



8. Is the sum of the measures of the angles of a triangle always 180°? Explain the apparent contradiction.
9. Locate any three points on the globe. Draw lines to join the three points. Measure the angles at each vertex. What is the sum of the measures? Compare your answer with those of your classmates. Can you explain what you observe?

Using the Library

1. Read the following pages in *Life Science Library: Mathematics*.

   Page 94 – Report on the Golden Ratio and Golden Rectangle.

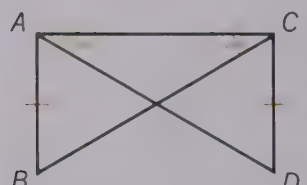
   Page 18 ff. – Report on Topology, Mobius Strip, Mazes and the Seven Bridges of Konigsberg.

2. Read the following pages in *The Wonderful World of Mathematics*.

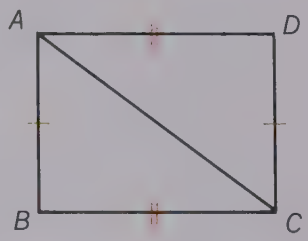
   Pages 12-13 and 20-21 – Report on the History of Units of Measures.

REVIEW EXERCISE 9

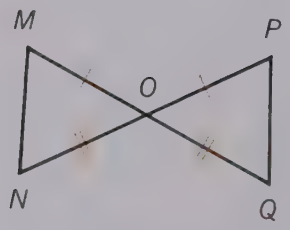
1. Tell whether or not the triangles in each pair are congruent. For those that are congruent tell which rule applies.



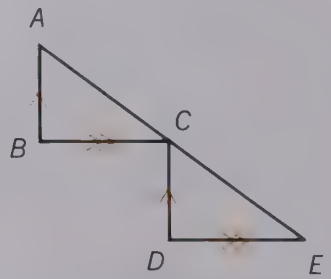
$\triangle ABC$? $\triangle CDA$?



$\triangle ABC$? $\triangle CDA$?



$\triangle MNO$? $\triangle PQO$?



$\overline{AB} \| \overline{DC}$
$\overline{BC} \| \overline{DE}$

$\triangle ABC$? $\triangle CDE$?

2. Is the set {10, 24, 25} a Pythagorean Triple?

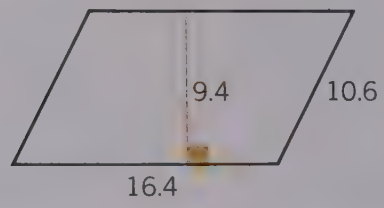
3. Two angles of a triangle have the measures of 35° and 65°. What is the measure of the third angle?

4. With compasses, straightedge and protractor construct an octagon.

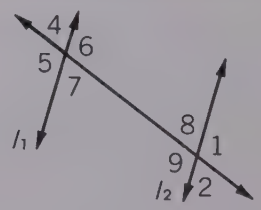
5. The measures of three angles of a triangle are represented by $3x - 9$, $3x + 12$, and $6x + 9$. Find the measures of the three angles of the triangle.

6. A parallelepiped has a base whose measure is 18 square inches and a height whose measure is 8.4 inches. What is the measure of the volume of the parallelepiped?

7. A parallelogram has the dimensions shown in the diagram. Find the measure of the area of the parallelogram.

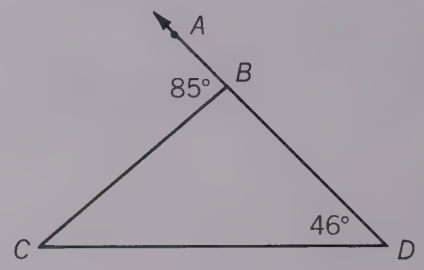


8. Lines l_1 and l_2 are parallel. Tell why each statement is true.

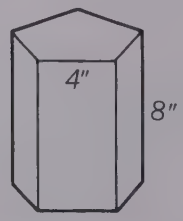


(a) $m\angle 4 = m\angle 8$
(b) $m\angle 8 = m\angle 2$
(c) $m\angle 6 = m\angle 9$
(d) $m\angle 4 = m\angle 2$
(e) $m\angle 4 + m\angle 6 = 180°$

9. What is the measure of $\angle BCD$ shown.



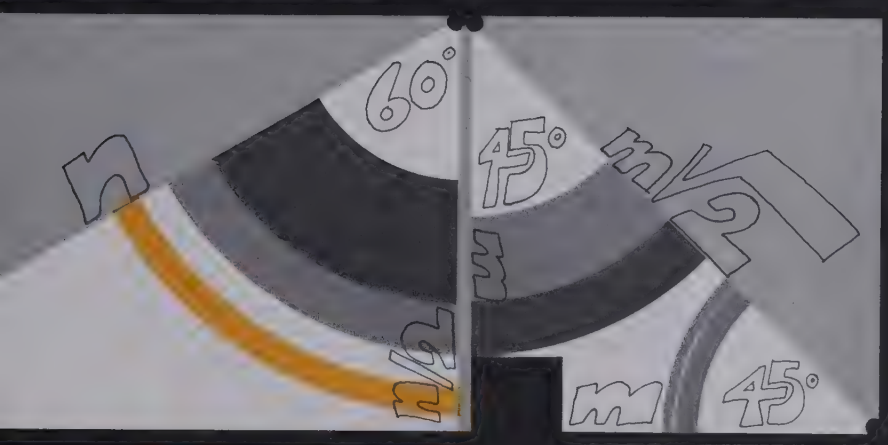
10. What is the measure of the volume of the pentagonal prism shown. (The base is regular.)

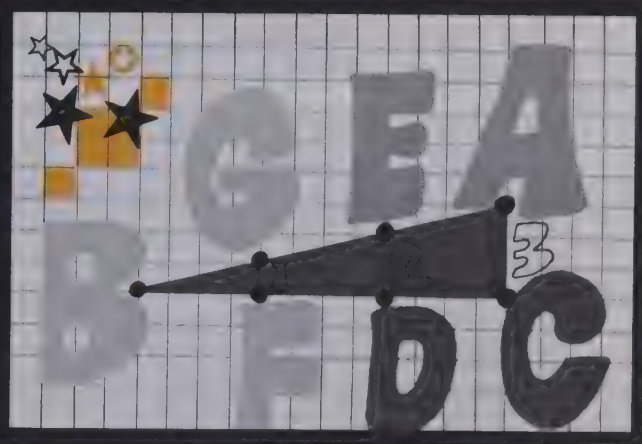


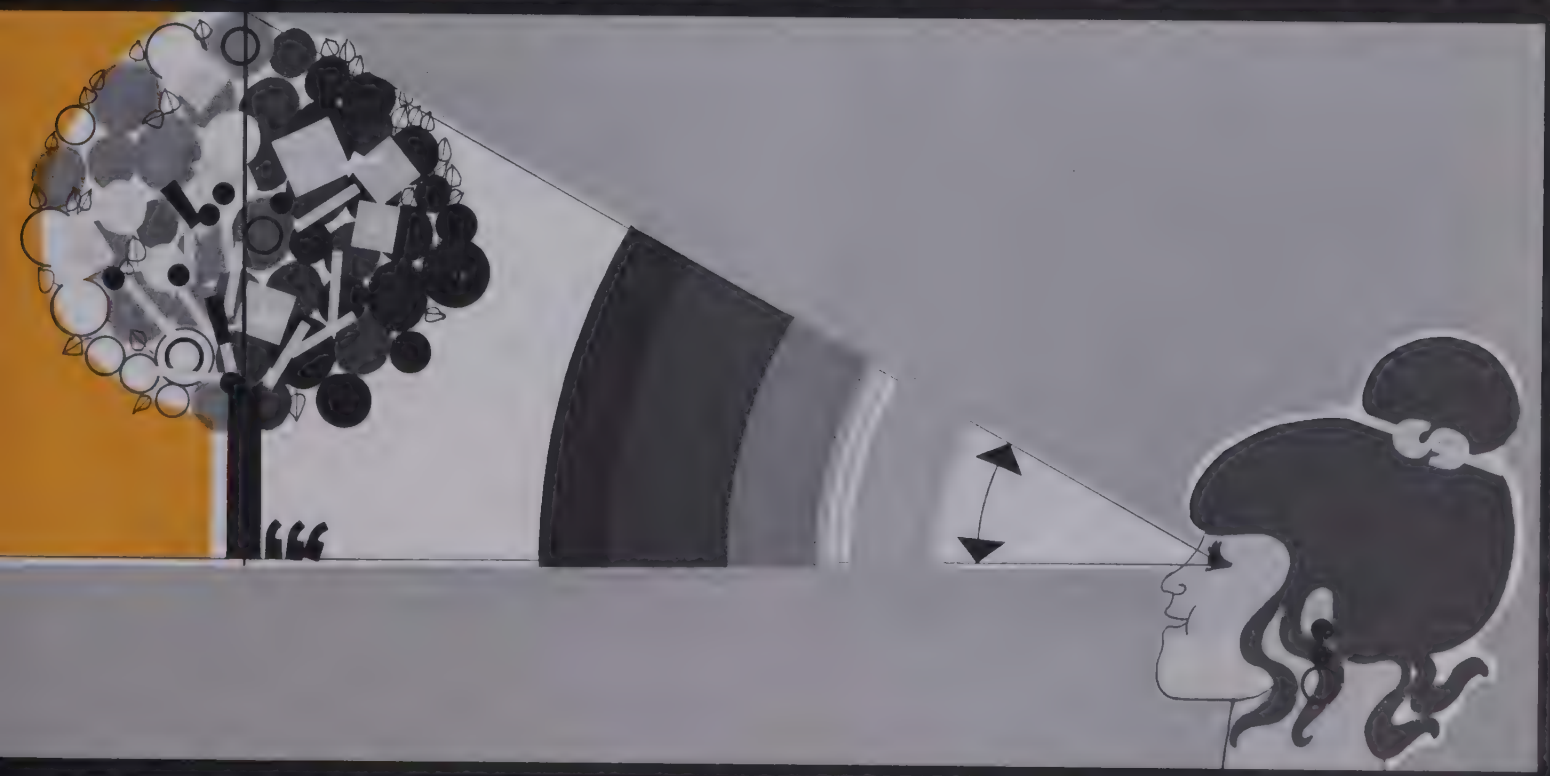
11. Draw any triangle ABC. Find the midpoints of each side and join with segments.
    (a) How many congruent triangles can you find?
    (b) How many parallel line segments can you find?

12. Write the converse of the statement: If a quadrilateral is a parallelogram then the opposite sides are parallel.

Chapter Ten
TRIGONOMETRY







10·1 Introduction

It is not known exactly when trigonometry was first used but some basic principles of trigonometry were stated by Thales about 600 B.C. Thales was presented with the problem of determining the height of a pyramid. He summed up the idea of similar triangles in this statement.

> The lengths of the sides of two triangles with equal angles are proportional.

Early astronomer, Hipparchus, in observatory at Alexandria.

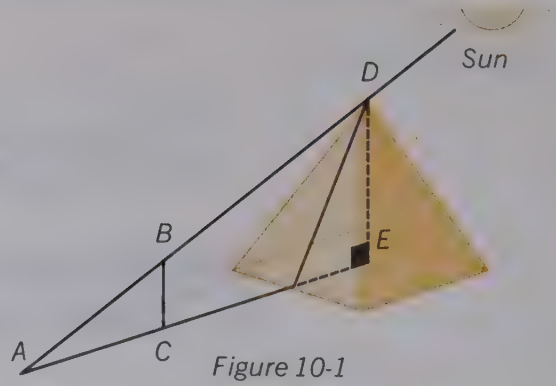
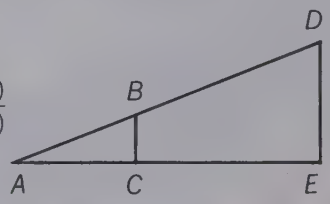


Figure 10-1

Let us review the use of this statement to find the height of the pyramid in Figure 10-1 given $m(\overline{AC}) = 10$ feet, $m(\overline{AE}) = 1000$ feet, and $m(\overline{BC}) = 4$ feet. (We realise that the two triangles, ABC and ADE have equal angles and are therefore similar.) Therefore,

$$\frac{m(\overline{AC})}{m(\overline{AE})} = \frac{m(\overline{BC})}{m(\overline{DE})} = \frac{m(\overline{AB})}{m(\overline{AD})}$$



Using the proportion in the box and making the appropriate substitutions we get

$$\frac{10}{1000} = \frac{4}{m(\overline{DE})}.$$

Solving we find the height of the pyramid to be 400 feet.

Other relations were known also. For example, if a ship were some distance from shore, this distance could be found. Study Figure 10-2 to see if you can understand the method the Greeks used over 2000 years ago.

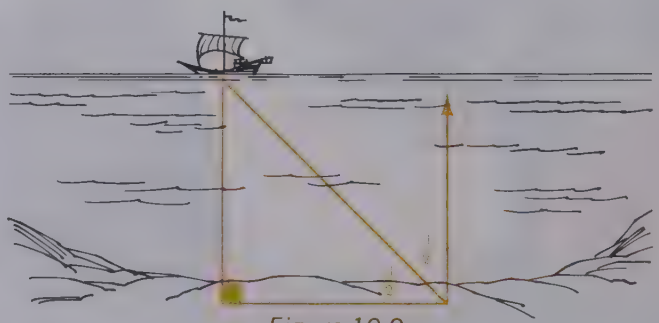


Figure 10-2

The ship was sighted at a right angle to the shore and at half a right angle to the shore. The distance between the two sighting points was the distance from the first sighting point to the ship. Why?

Eratosthenes calculated the circumference of the earth in the third century B.C. His calculations showed the earth to have a circumference of 25,000 nautical miles.

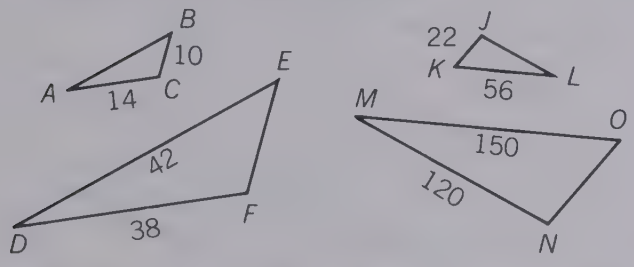
Hipparchus devised what we know today as the table of sines in the second century B.C. He contributed a great deal to trigonometry.

Trigonometry comes from the Greek words *trigonon* meaning triangle and *metron* meaning measure. Trigonometry is that branch of mathematics based on the relations between the sides and the angles of triangles.

Trigonometry is the basis of all surveying. In this chapter you will be doing some simple surveying.

EXERCISE 10·1

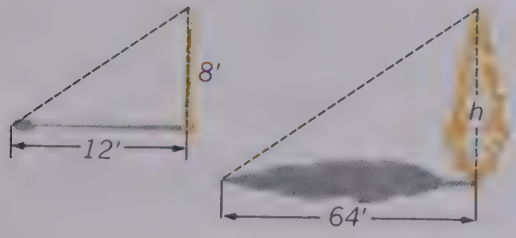
1. Find the measure of the indicated sides in each case. Use slide rule or calculator.



$\triangle ABC$ is similar to $\triangle DEF$. $\triangle JKL$ is similar to $\triangle NOM$.

(a) Find $m(\overline{EF})$.
(b) Find $m(\overline{AB})$.
(c) Find $m(\overline{NO})$.
(d) Find $m(\overline{JL})$.

2. The length of the shadow of a tree is 64 feet. At the same time the length of the shadow of an 8-foot lamp post nearby is 12 feet. Find the height of the tree.



Projects

Project 1

Purpose

To find the height of the flagpole in front of your school.

Method

Choose a sunny day to do this project.

1. Measure the length of the shadow cast by the flagpole.
2. Stand a yardstick vertically on the ground near the flagpole.
3. Measure the length of the shadow of the yardstick.
4. Use the relation of similar triangles to find the height of the flagpole.
5. Repeat 2, 3, and 4 several times but use students in place of the yardstick.
6. Average the results you have for the height of the flagpole.

10·2 Displacements

Figure 10-3 shows the route of a hockey puck as it is passed down the ice in the Montreal Forum.

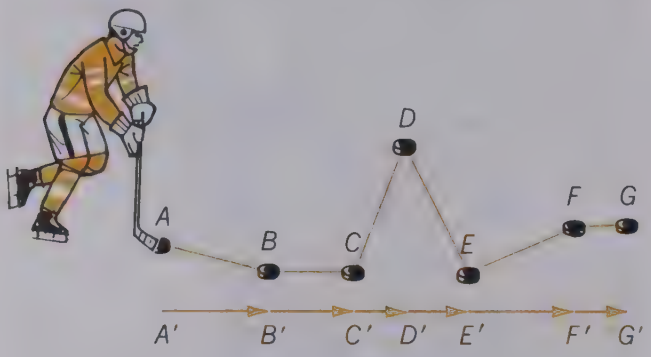


Figure 10-3

The object is to take the puck at A and put it in the goal at G. You will note the puck does not travel the shortest route but is moving forward all the time. (This does not always happen in a hockey game.) The straight line shows the forward displacements. In some cases the forward displacement of a pass is large while in other cases it is not. Consider the pass represented by $\overline{AB}$ in comparison to the pass represented by $\overline{DE}$. While the pass $\overline{DE}$ is a longer pass than $\overline{AB}$ the forward displacement of $\overline{DE}$ is not nearly as long as that of $\overline{AB}$.

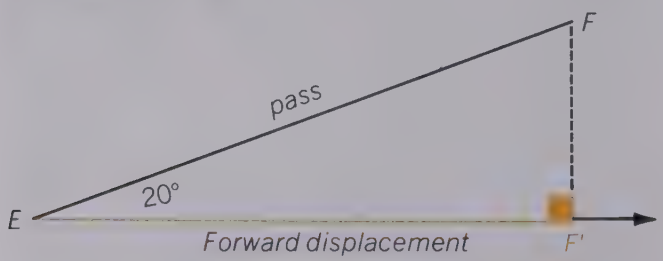


Figure 10-4

Let us consider the pass in Figure 10-4. The line of the pass $\overline{EF}$ forms an angle with the forward displacement $\overline{EF'}$. The measure of this angle is 20°. The length of the pass $\overline{EF}$ is 10 feet. From the scale drawing we find the forward displacement is 9.4 feet. From the study of similar triangles we know that if the length of the pass were 3 feet the forward displacement would be $\frac{3}{10} \times 9.4$ feet or 2.82 feet. If the length

of the pass were 1 foot then the forward displacement would be $\frac{1}{10} \times 9.4$ feet or 0.94 feet.

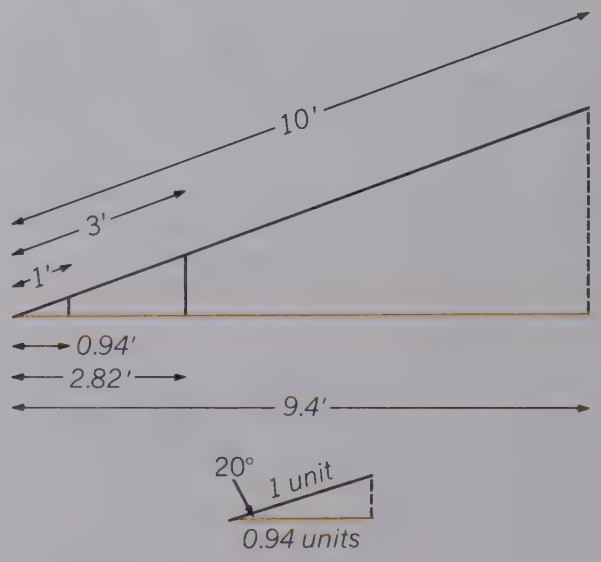


Figure 10-5

For an angle of 20° a pass of 1 unit results in 0.94 units forward displacement.

EXERCISE 10·2

In questions 1 to 4 the pass is at an angle of 20° to the forward displacement.

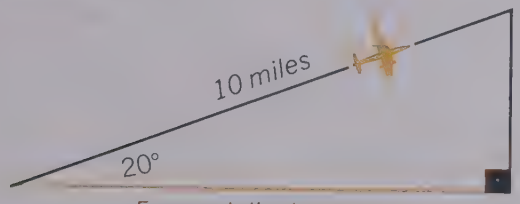
1. If the pass is 5 feet what is the forward displacement?

2. If the pass is 20 feet what is the forward displacement?

3. If the forward displacement is 4.7 feet what is the length of the pass?

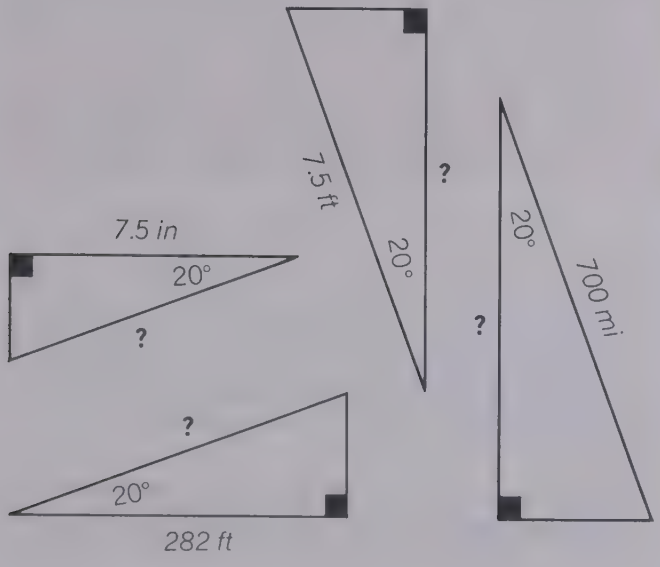
4. If the measure of the forward displacement is 3.76 feet what is the measure of the pass?

5. An airplane is climbing at an angle whose measure is 20° to the horizontal. If the distance the plane flies is 10 miles, what is the forward displacement?



Forward displacement

6. Find the measure of the distance indicated in each of the following figures.



10·3 Exploring

Since we will be talking about measures of angles and measures of lengths and since we wish to be brief in our explanations, we will, when there is no chance of ambiguity, speak of an angle whose measure is 20° as an angle of 20° and a distance whose measure is 40 feet as a distance of 40 feet, or a phrase similar to these. Remember though, we are speaking of measures.

Projects

Project 1

Purpose

To determine what happens to the measure of the forward displacement as the angle between the pass and the forward displacement increases.

Method

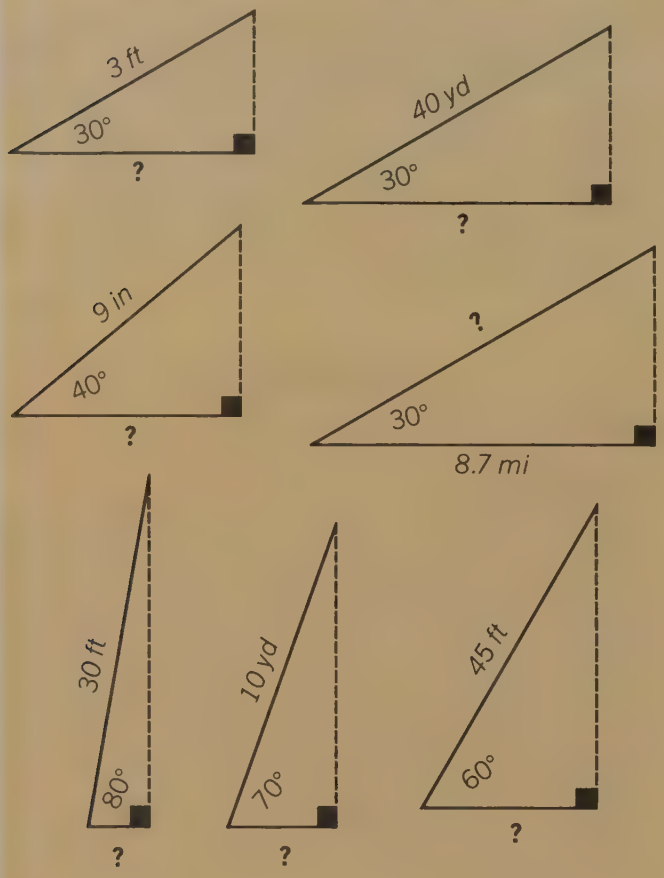
1. Draw accurately a right triangle in which the pass (hypotenuse) is 1 unit and the angle is 20°. Repeat for right triangles with an angle of 30°, 40°, 50°, 60°, 70°, and 80°. Choose your own unit.

2. Measure the forward displacements. Record the data in this table. These measures are approximate.

ANGLE	FORWARD DISPLACEMENT
20°	0.94
30°	
⋮	
70°	
80°	

3. What happens to the length of the forward displacement as the measure of the angle increases?

4. Use the information gained in 2 to calculate the distances required in the following diagrams.



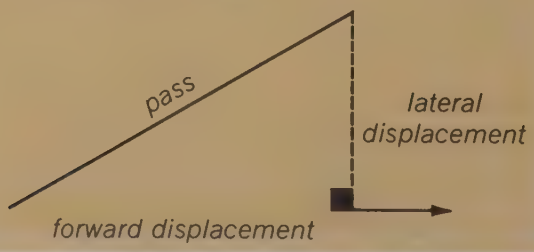
Project 2

Purpose

To determine what happens to the lateral displacement as the measure of the angle formed by the pass and the forward displacement increases.

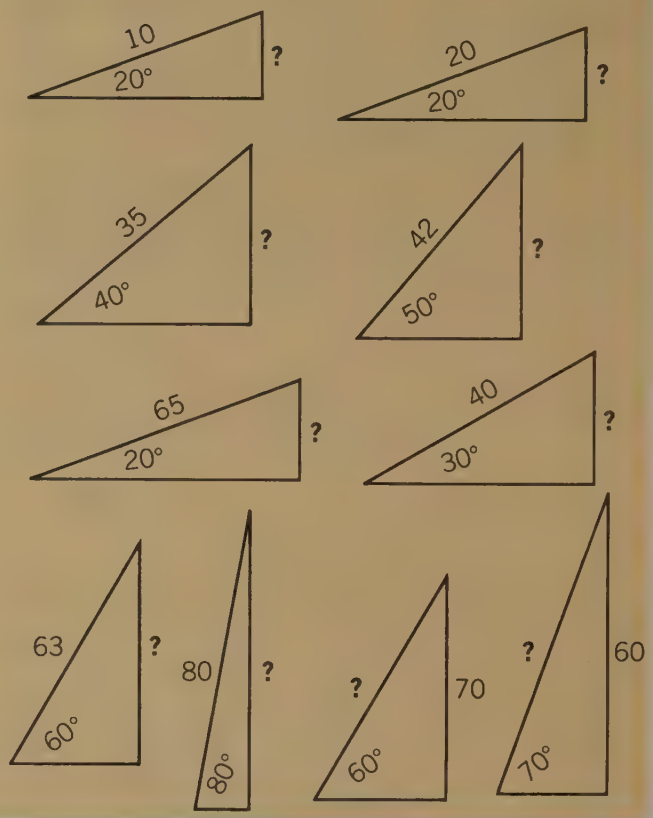
Background

The *lateral* (sideward) *displacement* is represented by the third side of the triangle.



Method

1. Use the diagrams from step 1 of Project 1 above.
2. Measure the lateral displacements. Record the data in a table as in Project 1.
3. What happens to the length of the lateral displacement as the angle described increases?
4. From the information in 2, calculate the required distances in each of the following diagrams.

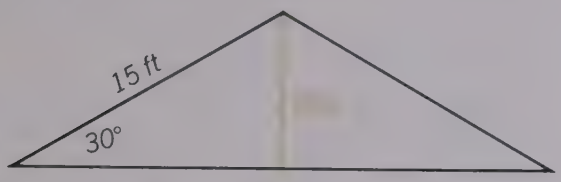


EXERCISE 10-3

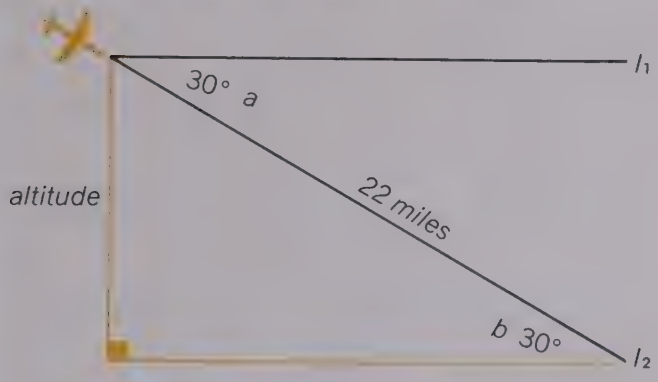
1. Use the information from the table in Project 1 to find the forward displacement in each triangle in step 4 of Project 2.
2. Use the information from the table in Project 2 to find the lateral displacement in each triangle in step 4 of Project 1.

Use the tables from Projects 1 and 2 to do the following questions.

3. Find the *rise* when the rafters are 15 feet long and the angle is 30°.



4. An airplane runs out of fuel and goes into a glide angle of 30° to the horizontal.
The distance of the glide is 22 miles.



(a) What is the forward displacement of his glide?
(b) What was the plane's altitude?
(c) Lines l_1 and l_2 are both horizontal lines. Are horizontal lines parallel? How do we know $\angle a = \angle b$?

10·4 Sine and Cosine

In section 10·3 we have seen that the forward displacement is a function of the angle since the measure of the pass was a constant 1 unit, and that the lateral displacement is a different function of the angle. The functions are readily seen if the data are shown as mappings. The data in the two tables that follow are taken from the tables in Projects 1 and 2 in section 10·3. The data are approximate.

Table 1

ANGLE	FORWARD DISPLACEMENT	ANGLE	LATERAL DISPLACEMENT
20°	→ 0.94	20°	→ 0.34
30°	→ 0.87	30°	→ 0.50
40°	→ 0.77	40°	→ 0.64
50°	→ 0.64	50°	→ 0.77
60°	→ 0.50	60°	→ 0.87
70°	→ 0.34	70°	→ 0.94
80°	→ 0.17	80°	→ 0.98

Mathematicians have recognized the importance of these functions that occur in the right triangle. The pass is called the *hypotenuse*, the forward displacement is called the *cosine* and the lateral displacement is called the *sine* (when the pass is 1 unit). The forward displacement for a unit hypotenuse at the angle of $\theta°$ (theta degrees) is *cosine* θ. The lateral displacement for a unit hypotenuse at an angle of $\theta°$ is *sine* $\theta°$. The two abbreviations are cos θ and $\theta°$. θ is a variable much like x, customarily used for the measure of an angle. Sometimes other Greek letters are used such as ϕ (phi) or α (alpha). We can now write the mappings in Table 1 as shown in Table 2. We will not use arrows, but remember, they are mappings.

Table 2

angle $\theta°$	sin $\theta°$	cos $\theta°$
20°	0.34	0.94
30°	0.50	0.87
40°	0.64	0.77
50°	0.77	0.64
60°	0.87	0.50
70°	0.94	0.34
80°	0.98	0.17

Study Table 2. What relation do you find between sin $\theta°$ and cos $\theta°$? What do you think is sin 10°? What do you think is cos 10°?

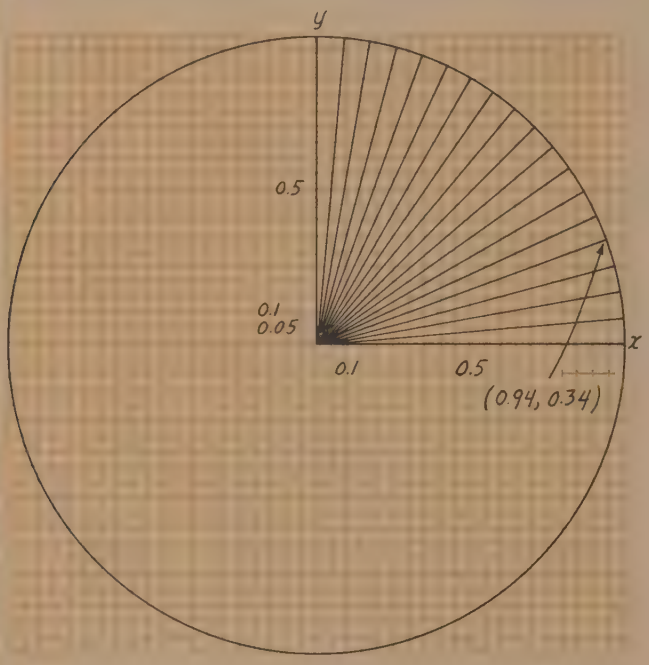
Project

Project 1

Purpose

To determine a table for sin $\theta°$ and cos $\theta°$ by using a unit circle.

Method

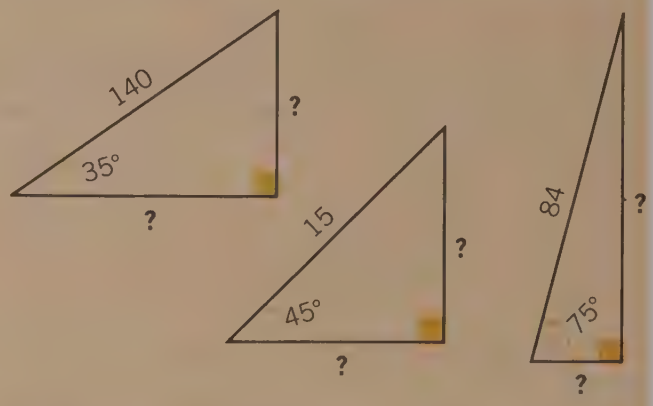


1. Use a sheet of graph paper ruled 10 lines to the inch. Draw a circle of radius 2 inches. Be as exact as possible.
2. Draw the x-axis and y-axis to intersect in the centre of the circle. See the diagram above.
3. Consider the radius to be 1 unit. Mark the scale on the axes (each line = 0.05 units).
4. Draw angles with the following measures with the vertex at the centre and the x-axis as one arm. Terminate the second arm in the circle in the first quadrant. These angles are now in *standard position*:
   10°, 20°, 30°, 40°, 50°, 60°, 70°, 80°.
5. On the graph paper near the endpoint of the second arm, write coordinates of the point in which the arm intersects the circle.
6. Compare the ordered pairs in 5 with Table 2 above.
7. Do you agree that the ordered pairs are in general representative of (cos $\theta°$, sin $\theta°$)?
8. Repeat steps 4 and 5 for angles that have measures 5°, 15°, . . . , 85°.
9. Record the data from steps 5 and 8 in a table. The numbers in the cos $\theta°$ and sin $\theta°$ columns are approximate.

θ	sin $\theta°$	cos $\theta°$
5		
10		
15		
20	0.34	0.94
.		
.		
.		
85		

It is customary in trigonometric tables to list the sin $\theta°$ values before cos $\theta°$.

10. Find the measure of the sides of the triangles.



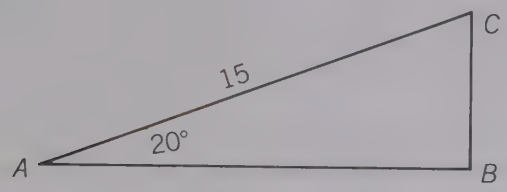
11. Refer to 9. As the angle $\theta°$ increases what happens to the value of cos $\theta°$? of sin $\theta°$?

The values recorded in the table in Project 1 are approximate values. Tables have been made which are more accurate. For tables of sin $\theta°$ and cos $\theta°$ refer to page 269. Compare the values in this text with those you found in Project 1.

Let us now formally state how to use sin $\theta°$ and cos $\theta°$ tables. In the general case we will represent measures of the three sides of the triangle by r, x, and y as in the diagram.

Specific Case

Given



Triangle ABC with $m(\overline{AC}) = 15$ and $m \angle A = 20°$

Required

To find $m(\overline{AB})$ and $m(\overline{BC})$

Solution

To find $m(\overline{AB})$ we multiply the value of cos 20° by 15.

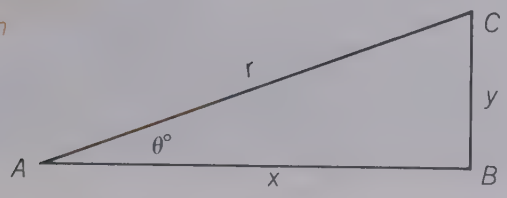
$$m(\overline{AB}) = 0.94 \times 15$$
$$= 14.1$$

To find $m(\overline{BC})$ we multiply sin 20° by 15.

$$m(\overline{BC}) = 0.34 \times 15$$
$$= 5.1$$

General Case

Given



Triangle ABC with $m(\overline{AC}) = r$ and $m \angle A = \theta°$

Required

To find $m(\overline{AB})$ and $m(\overline{BC})$

Solution

To find $m(\overline{AB})$ which we call x we multiply cos $\theta°$ by r.

$$x = r \cdot \cos \theta°$$

To find $m(\overline{BC})$ which we call y we multiply sin $\theta°$ by r.

$$y = r \cdot \sin \theta°$$

Example 1

Find θ when (a) $\cos \theta° = 0.7771$
(b) $\sin \theta° = 0.6561$.

Solution

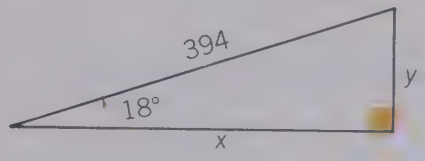
(a) Using the tables in this text we look in the column marked 'cos θ' until we find 0.7771. We then look across to the column marked θ and we find '39', $\therefore \theta = 39$.

θ	sin $\theta°$	cos $\theta°$
0	0.0000	1.0000
1	.0175	0.9998
⋮	⋮	⋮
38	0.6157	0.7880
39	0.6293	0.7771
40	0.6428	0.7660
41	0.6561	0.7547

(b) We go down the sin $\theta°$ column until we come to 0.6561. We go across to the $\theta°$ column to find $\theta = 41$.

Example 2

Find the lengths of the sides x and y.



Solution

We know that $x = r \bullet \cos \theta°$ when r is the length of the hypotenuse and x is the side adjacent to the angle.

$$x = r \bullet \cos \theta°$$
$$x = 394 \times \cos 18° \quad \text{(Look in tables for cos 18°)}$$
$$x = 394 \times 0.9511$$
$$x = 374.7$$

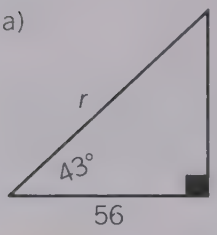
Similarly, $x = r \bullet \sin \theta°$

$$y = r \times \sin 18° \quad \text{(Look in tables for sin 18°)}$$
$$y = 394 \times 0.3090$$
$$y = 121.7$$

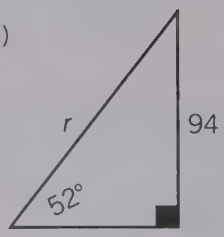
Example 3

Find the hypotenuse (r) given.

(a)



(b)



Solution

(a) We know $x = r \cos \theta°$

$$\therefore r = \frac{x}{\cos \theta°}$$

$$r = \frac{56}{\cos 43°} \text{ (make replacements)}$$

$$r = \frac{56}{0.7314} \text{ (use tables)}$$

$$r = 76.57$$

(b) Similarly, $y = r \sin \theta°$

$$\therefore r = \frac{y}{\sin \theta°}$$

$$r = \frac{94}{\sin 52°} \text{ (make replacements)}$$

$$r = \frac{94}{0.7880} \text{ (use tables)}$$

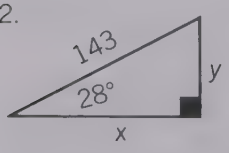
$$r = 119.3$$

EXERCISE 10·4

1. Use tables to find the value of each:
   (a) cos 33° (b) sin 47°
   (c) cos 56° (d) sin 72°
   (e) cos 5° (f) sin 60°
   (g) cos 82° (h) sin 68°
   (i) sin 12°.

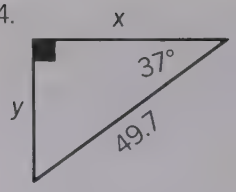
Use the tables in the text. Find the unknown sides. Use calculators if available.

2.

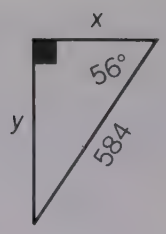


3.

4.



5.



6. Find the value of θ when
   (a) $\cos \theta° = 0.9877$ (b) $\sin \theta° = 0.8910$
   (c) $\cos \theta° = 0.8746$ (d) $\sin \theta° = 0.4067$
   (e) $\cos \theta° = 0.2419$ (f) $\sin \theta° = 0.9511$.

7. Solve $x = r \cdot \cos \theta°$ for $\cos \theta°$.
   Solve $y = r \cdot \sin \theta°$ for $\sin \theta°$.

8. Make up one question (with the answer) for your classmates. Write it on the chalkboard. Do the questions your classmates make up.

10·5 Tangent

There are several ways in which we can refer to the sides of a right triangle. We have used several already. When the triangle is in the standard position we use the symbols r, x, and y for the sides. Another that we use in trigonometry is introduced here. See Figure 10-6. We identify an angle called θ. The hypotenuse is named as before. The one leg that forms part of the angle is called the *adjacent* side (meaning *next to*). The side opposite the angle is called the *opposite* side.

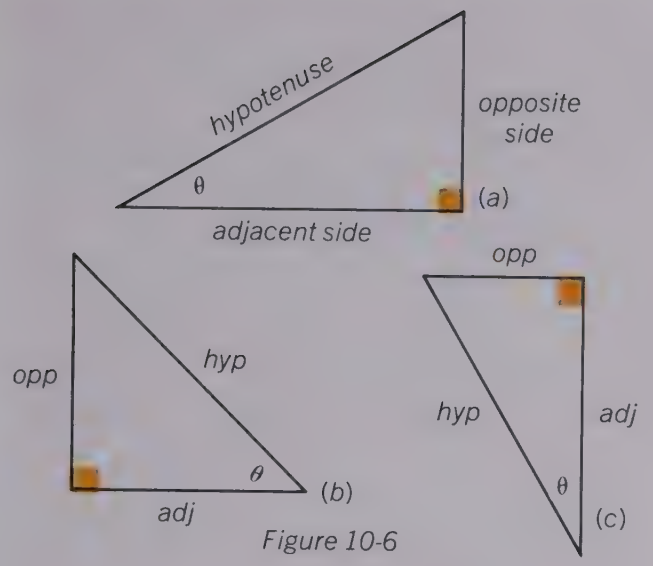
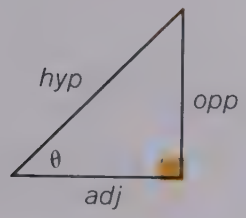


Figure 10-6

In discussing the ratios in the next sections, for the sake of brevity, we shall use the term opposite (abbreviated *opp*) to mean the measure of the opposite side, the term hypotenuse (*hyp*) to mean the measure of the hypotenuse, and adjacent (*adj*) to mean the measure of the adjacent side.

We have been working with the ratios of the measures of the two legs to the measure of the hypotenuse of a right triangle, specifically the cosine and sine ratios. We can consider the cosine $\theta°$ ratio as the ratio of the adjacent side to the hypotenuse (we mean the measures of these):

$$\cos \theta° = \frac{\text{adj}}{\text{hyp}}$$



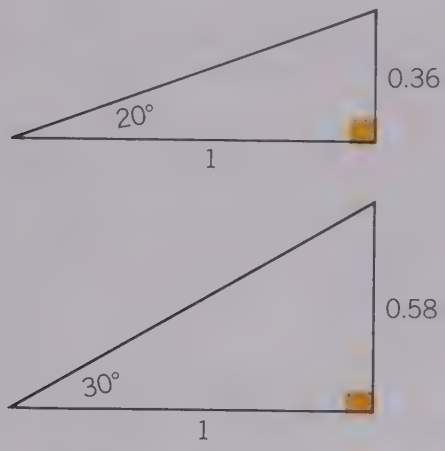
Previously in a number of examples we assigned the hypotenuse a value of one unit so that

$$\cos \theta° = \text{adj}.$$

Similarly the sine $\theta°$ ratio is the ratio of the opposite side to the hypotenuse.

$$\sin \theta \quad \frac{\text{opp}}{\text{hyp}}$$

Let us now assign a unit value to the adjacent side and determine the length of the opposite side for various values of θ.

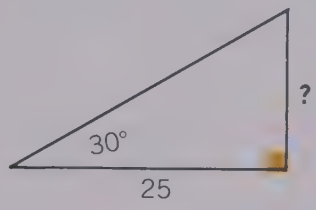


The actual values (by measurement) of the opposite sides are:

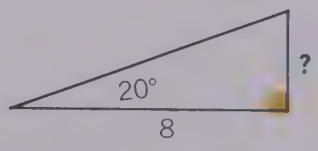
$$20° \rightarrow 0.36 \qquad 30° \rightarrow 0.58.$$

Can you find the lengths of the opposite sides in these triangles?

(a)



(b)



Did you get 2.88, and 14.5 respectively?

Tangent θ is the length of the opposite side in a right triangle when the length of the adjacent side is 1 unit. Therefore, tangent 20° (abbreviated tan 20°) is 0.36 and tan 30° is 0.58. The opposite side in each is:

(a) opposite side $= 8 \cdot \tan 20°$
$= 8 \times 0.36$
$= 2.88$

(b) opposite side $= 25 \cdot \tan 30°$
$= 25 \times 0.58$
$= 14.5$

Project

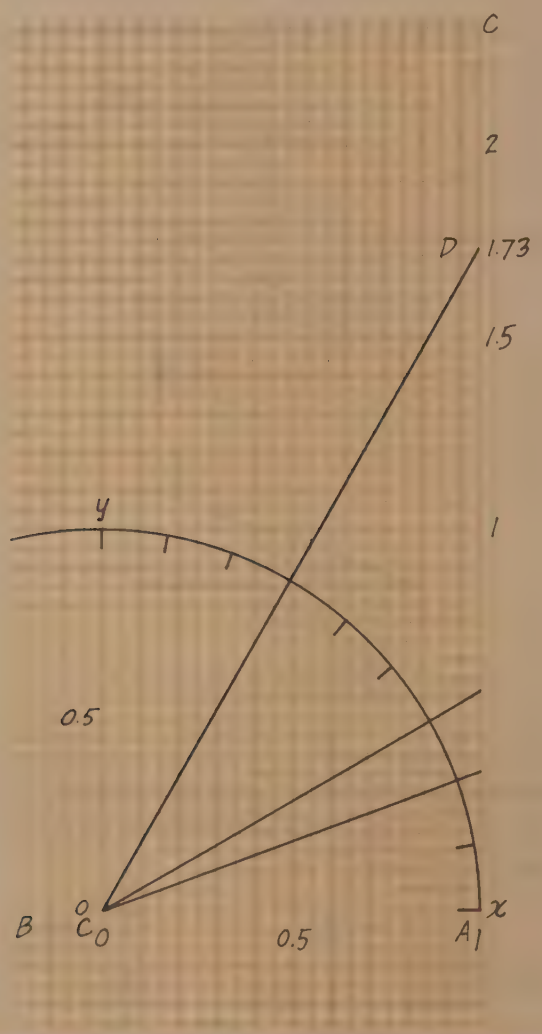
Project 1

Purpose

To use the unit circle to obtain a table of tangent $\theta°$ values.

Method

1. On a sheet of graph paper draw a unit circle with the origin of the set of axes as the centre. Use the same scale as in Project 1 of Section 10·4. Locate the circle at the bottom of a full page graph paper.
2. Label the x-axis *BOA* as shown.
3. Perpendicular to $\overline{BA}$ draw line $\overleftrightarrow{AC}$.



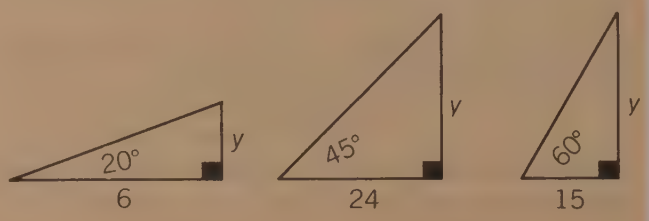
4. Mark on $\overrightarrow{AC}$ the scale starting with zero at *A*.
5. Draw $\angle DOA$ in standard position where $m\angle DOA = 60°$ and *D* is on $\overrightarrow{AC}$.
6. From the scale determine the measure of $\overline{AD}$. This measure is tangent 60°. Tangent 60=1.73. Label *D* with the ordinate value.
7. Repeat 5 and 6 for angle measures from 5° to 75° in multiples of 5.
8. Record the data in a table.

$\theta°$	tan $\theta°$
5°	
10°	
•	
•	
•	
60°	1.73
•	
•	
•	
75°	

9. Compare this table with the tangent $\theta°$ table in this book.
10. Attach a piece of graph paper to the top of the sheet you have been working with. Draw $\angle DOA$ with measures 80°, 85° and as close to 90° as your paper allows. Record the value of tan $\theta°$

tan 90° is undefined.

11. What happens to the value of tan $\theta°$ as $\theta°$ increases beyond 75° (but remains less than 90°)?
12. Use the table in 8 to find the opposite sides:

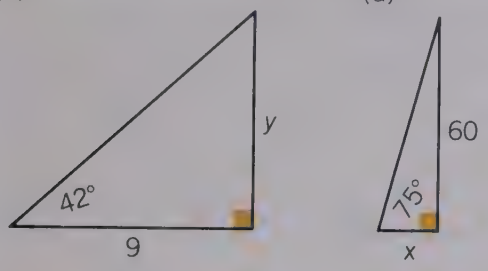


13. What do you think is the value of tan 0°?

Example 1

Use the tables in this book to find:
(a) tan 42° (b) tan 75°.

Then find the opposite side in each triangle.
(c) (d)



Solution

(a) From the table we find tan 42° = 0.9004. (Notice we use the equal sign but the values in the trigonometric tables usually are not exact; they are only approximate numbers. Only a few are exact.)

(b) tan 75° = 3.7321.

θ	sin θ°	cos θ°	tan θ°
⋮	⋮	⋮	⋮
41			0.8693
42			0.9004
43			0.9325
⋮			⋮
74			3.4874
75			3.7321
76			4.0108

(c) Opposite side = adjacent side × tan 42°.
Since tan 42° = 0.9004
then opposite side = 9 × 0.9004.
Therefore $y = 8.1036$

(d) Opposite side = adjacent side × tan 42°.
Since tan 75° = 3.7321
then $60 = x(3.7321)$

$$x = \frac{60}{3.7321}$$

$$x = 16.08$$

Example 2

Find θ when tan θ° = 1.1726.

Solution

We look in the table of tangents to find 1.1726.

θ	sin θ°	cos θ°	tan θ°
⋮			⋮
48			1.1106
49			1.1504
50			1.1918

We know $1.1504 < 1.1726 < 1.1918$
Therefore $49 < \theta < 50$

1.1504
1.1726 } difference of 222
1.1918 } difference of 192

Hence we see that 1.1726 is closer to 1.1918 than to 1.1504, therefore θ is closer to 50 than to 49. Our tables give only whole number values of θ. Some tables are more accurate. Therefore, θ is 50°.

Example 3

Find θ when cos θ° = 0.400

Solution

We look in the table of cosines to find 0.4000.

θ	sin θ°	cos θ°	tan θ°
⋮		⋮	
65		.4226	
66		.4067	
67		.3907	

Since $0.4067 < 0.4000 < 0.3907$
then $66 < \theta < 67$

0.4067
0.4000 } difference of 67
0.3907 } difference of 93

Since θ° is closer to 66° than to 67°, we will use 66° for θ°.

EXERCISE 10·5

1. What is the range of sine θ when the domain is $0° \leq \theta \leq 90°$?

2. What is the range of cosine θ when the domain is $0° \leq \theta \leq 90°$?

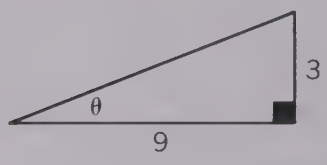
3. What is the value of tan 0°? of tan 89°? Does tan 90° have a value?

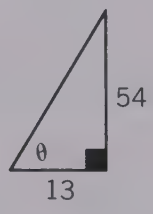
4. The cosine θ ratio is (always, sometimes, never) less than 1 and (always, sometimes, never) greater than 1.

5. The sine θ ratio is (always, sometimes, never) less than 1 and (always, sometimes, never) greater than 1.

6. The tangent θ is (always, sometimes, never) less than 1 and (always, sometimes, never) greater than 1.

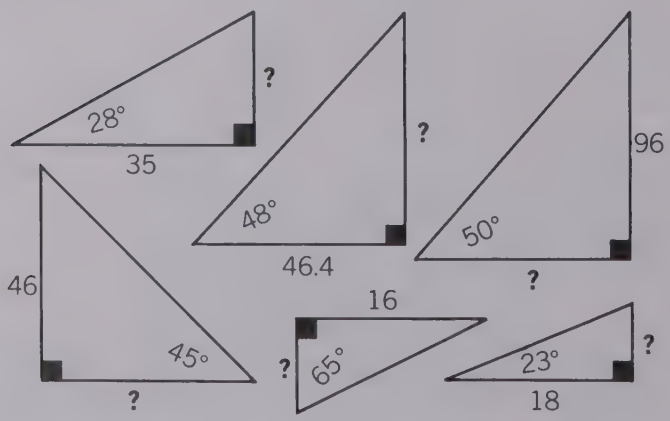
7. If opp = 3, adj = 9, what is tan θ? What is θ?





8. If opp = 54, adj = 13, what is tan θ? What is θ?

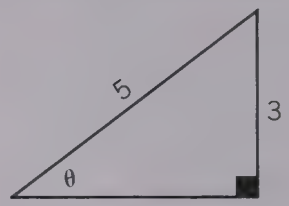
9. Find the measure of the side labelled **?**.

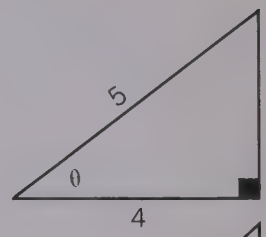


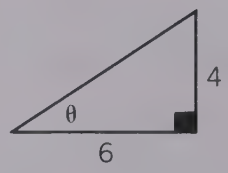
10. Find θ when cos θ =
    (a) 0.9000 (b) 0.7000 (c) 0.3000.

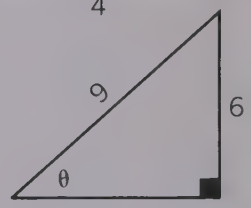
11. Find θ when sin θ =
    (a) 0.9000 (b) 0.7000 (c) 0.300.

12. Find θ in each case:

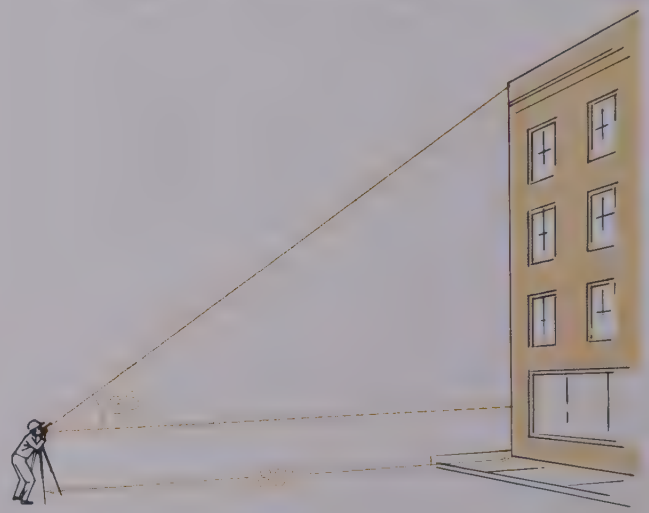








13. An engineer, using an instrument that is 5 feet off the ground, measures the angle of inclination of the top of a building which is 80 feet from him. The angle is 35°. Find the height of the building.

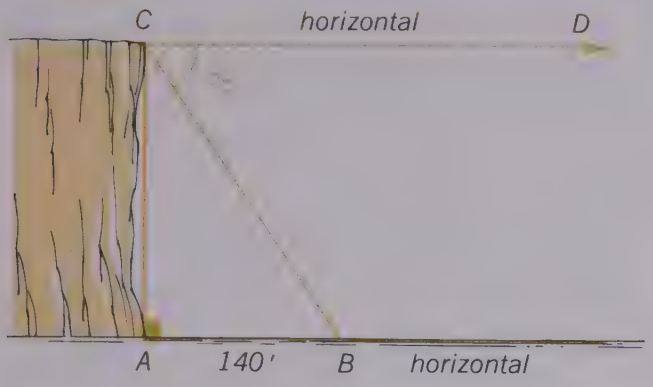


14. A man, at *B*, stands on the level 65 feet from a spot *A* which is directly below a point *C* on a cliff. The angle of inclination of the point *C* on the cliff is 74°. Find the height of the cliff.

15. A man, at *C* stands on a cliff overlooking the ocean. The *angle of depression* of the water's edge *B* is 56°. The water's edge is 140 feet from a

spot A directly below him. Point A is approximately the same level as the water's edge. Find the height of the cliff.

Hint! $\overline{AB} \parallel \overline{CD}$. Note the meaning of angle of depression



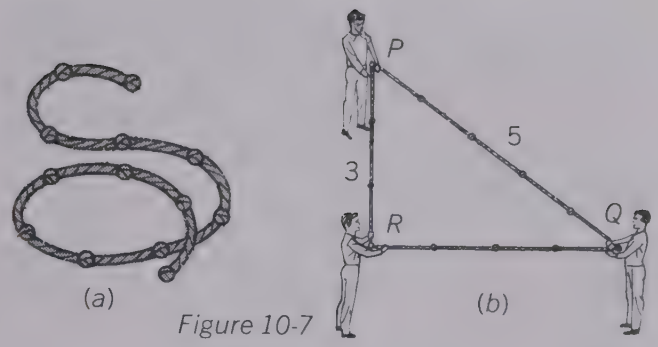


Figure 10-7

One of the earliest surveying instruments was a rope with a total of 13 knots equally spaced (Figure 10-7(a)). By stretching this rope, 3 men, one at each of P, Q and R, were able to construct a right angle in order to lay out land boundaries or the base of a pyramid. You may apply this same principle by using a tape (100 ft.) in the manner indicated in Figure 10-8. The angle at Q is a right angle.

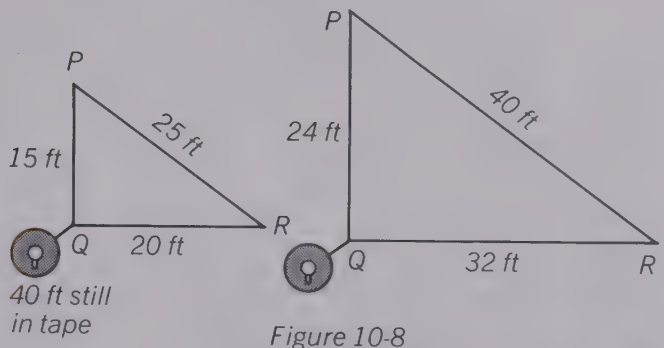


Figure 10-8

10·6 Simple Measurements

Since geometry originated with actual measurements let us now study some of the instruments and procedures that can be used to make geometric measurements.

The radar receiver shown on the right is used to detect formation of hailstorms in Alberta. The screen on the left shows the direction and the distance of all objects detected. Invite the radar division of your local police department to explain and illustrate how radar is used to detect and identify speeding motorists.

A useful way to determine direction is easy to illustrate. In Project 1 we illustrate how to find a meridian [redacted] sometimes called a line of longitude, and a line of latitude (the line perpendicular to the meridian).

Projects

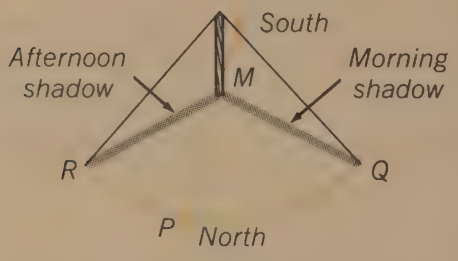
Project 1

Purpose

To determine a meridian and the line of latitude.

Method

1. Before noon, insert in the ground in a level area a vertical stick about 5 feet long (M).



2. Locate the shadow of its tip (Q). Draw a circle with radius $\overline{MQ}$ by means of a cord or measuring tape.
3. Later, in the afternoon, the shadow will again reach this circle (at R). Mark point R.
4. Draw $\angle RMQ$.
5. Bisect $\angle RMQ$. The meridian will be indicated by $\overleftrightarrow{MP}$.
6. Construct a line perpendicular to $\overleftrightarrow{MP}$ by using a tape measure as in Figure 10-8. This line is the line of latitude.
7. Obtain a magnetic compass and note the direction between the magnetic north line and the geographic north line.

It is important at this point to make a few comments about field exercises and the equipment for field exercises. The degree of accuracy of your measurements (direct and indirect) will depend on

1. accuracy of your instruments—how carefully they are made.
2. accuracy of your data—this depends on how carefully you take readings and handle equipment.

It is often necessary to determine distances that cannot be measured directly. We will show the construction and use of some simple instruments.

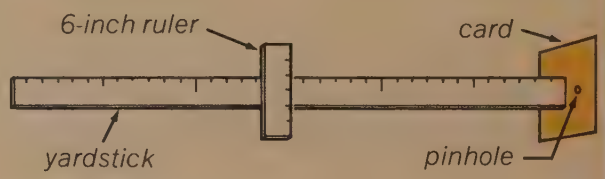
Project 2

Purpose

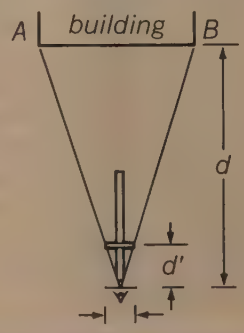
To make and use a simple cross-staff.

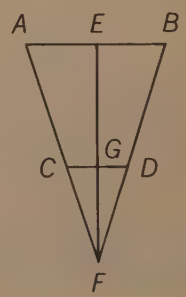
Method

1. Obtain a yardstick, 6-inch ruler and 4" × 6" index card. Assemble as shown. Thumbtack the card to the yardstick. Put a pinhole in the middle of the card.



3. Use the cross-staff to find the length of your school. From near the middle of the wall of the building walk directly away (perpendicular to wall) a known number of feet (e.g., 100 feet).
4. Hold the cross-staff to your eye such that you are sighting through the hole in the card. Slide the 6-inch ruler forward or back until the ruler just covers the length of the building.
5. See the diagram. Record the distance from the pinhole to the 6-inch ruler.





6. To calculate the distance from A to B, the length of the building, refer to the preceding diagram. $\triangle ABF$ is similar to $\triangle CDF$.
Therefore

$$\frac{m(\overline{AB})}{m(\overline{EF})} = \frac{m(\overline{CD})}{m(\overline{GF})}$$

$m(\overline{AB})$ represents the length of the building
$m(\overline{EF})$ represents the distance from the building (d).
$m(\overline{CD})$ represents the length of the 6-inch ruler ($\frac{1}{2}$ ft.).
$m(\overline{GF})$ represents the distance from the pinhole to the 6-inch ruler (d').

Therefore, the length of the building is

$$L = \frac{\frac{1}{2}d}{d'} \text{ (all measurements in feet)}$$

$$L = \frac{d}{2d'}.$$

If a 6-inch ruler is not used the formula is different.

7. Find the length of your school by means of the cross-staff.
8. By actual measurement find the length of the school.
9. Find the percentage error. Percentage error is the ratio (expressed in percent) of the error to the actual (true) value.

$$\text{Percentage error} = \frac{\text{error}}{\text{true value}} \times 100\%$$

10. Find the lengths of other buildings, or the length of the school grounds, or the width of the street, or the distance between two light standards.

The cross-staff may be improved upon and may be used in situations where it is not possible to find the distance the viewer is from the object being measured.

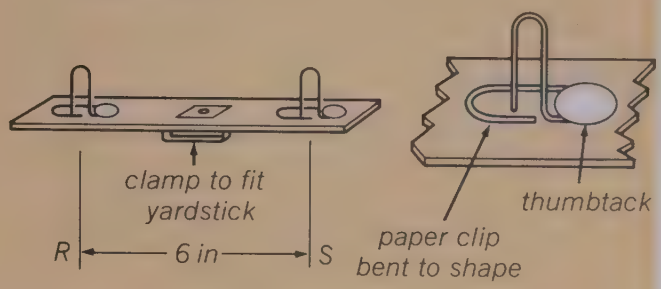
Project 3

Purpose

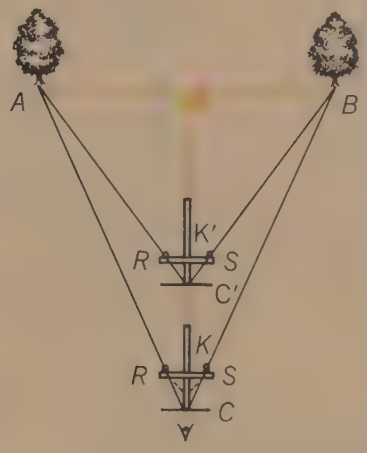
To construct an improved cross-staff and use it to measure the length of a building.

Method

1. The cross-arm (6-inch ruler) is replaced by the following piece.



2. In this case we need not know the distance the viewer is from the object. Take a sighting as before in Project 2. Record the distance from the cross-arm to the eyepiece.
3. Move further from the object (by about 10 feet or more) and take another sighting. Record the distance from the crosspiece to the eyepiece.



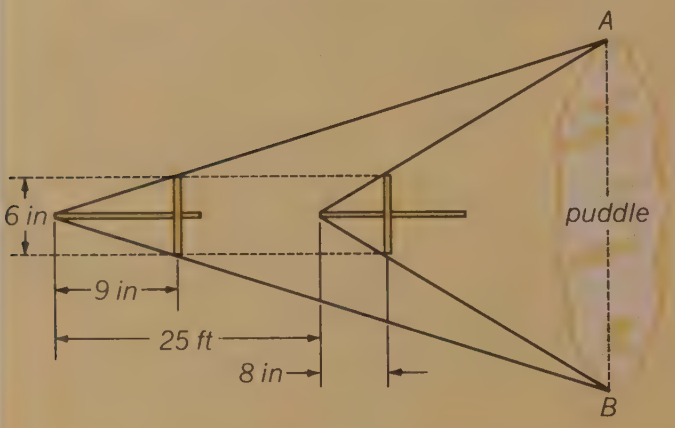
4. The calculation of the distance AB is based on similar triangles. The derivation of the formula is not included here.

$$m(\overline{AB}) = \frac{m(\overline{CC'}) \cdot m(\overline{RS})}{m(\overline{CK}) - m(\overline{C'K'})}$$

5. Find the length of your school building using this method.
6. Find the percentage error.
7. Is this error greater or less than the error in Project 2?

8. We are now able to find the distance between two points which are an unknown distance from the viewer. Note the line through the two points A and B must be perpendicular to the line through the points C and C'. (See the diagram above.)

9. Find the distance $\overline{AB}$ across the puddle.



10. Use this instrument to find distances assigned by your teacher.

Another early instrument was the astrolabe. It did the work of a number of instruments. It is easy to make and easy to use.

Project 4

Purpose

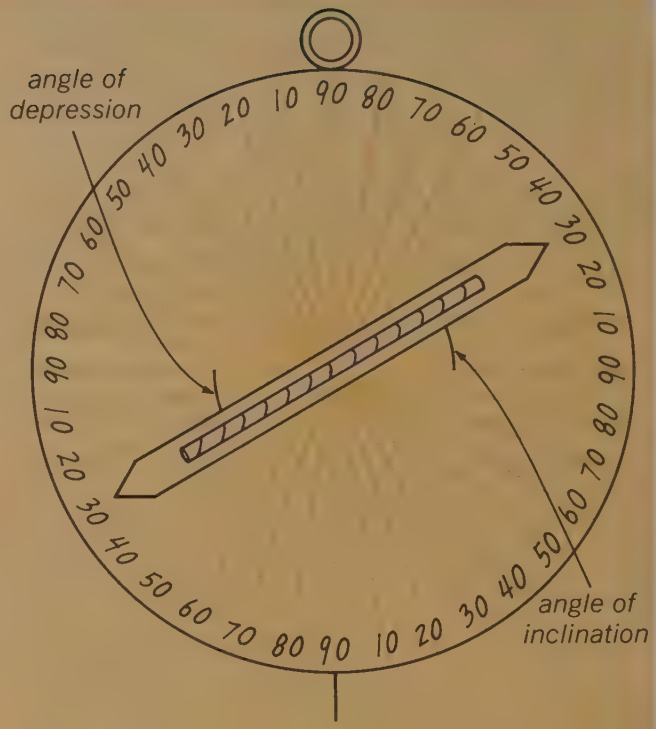
To make and use an astrolabe.

Method

1. From heavy cardboard or light fibreboard, cut a circle with radius 7 inches. (The size is not relevant but the larger the instrument is the more accurate your results will likely be.)

2. Attach a sighting arm by means of a paper fastener or small bolt so that the arm is free to rotate when a light touch is applied.

3. Exactly over the fastener, (the point of rotation) and in line with the points on the sighting arm, glue a drinking straw. Exercise care in this.



4. On the circular board attach a ring for suspending the instrument.

5. Suspend the instrument by this ring. Attach (temporarily) a string to the ring. To the other end of the string attach a weight. Let the weight hang free. Mark on the board the line indicated by the string. This is the vertical line. This line should go through the centre of the circle. Remove the string.

6. Draw a line perpendicular to the vertical line through the centre of the instrument. This is the horizontal line.

7. Starting with the horizontal line as zero, mark on the degree measures as shown in the above diagram. The degrees should be marked in units.

8. Now we are ready to use the astrolabe. It is used to measure angles of inclination and depression.

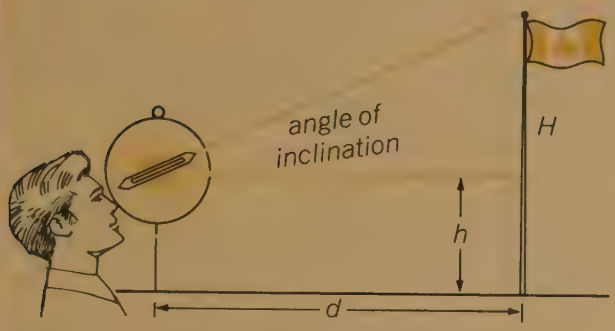
9. Stand a known distance (40 feet) from the foot of a flagpole (or tree). Hold the instrument in front of one eye so that you can sight through the straw. Move the sighting arm so that you can see the top of the flagpole.

Record:

(a) the angle of inclination;

(b) the height the middle of the instrument is above ground level (h);

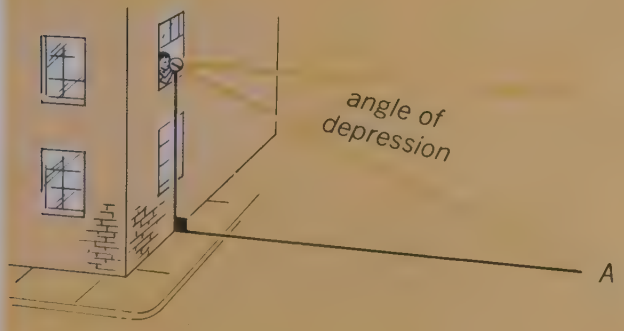
(c) the distance the instrument is from the flagpole (d).



10. Calculate the measure of H by using tan 0° function and d. Then add H and h to get the total height of the flagpole.

11. Use this instrument to find the height of your school building, TV tower, tree, high rise apartment building, etc.

12. Make up a question for your classmates which will involve the use of the astrolabe. Work in groups to answer the questions developed by the class.



The next instrument is one that is very useful and easy to use. It gives the answers automatically. It is used to find heights of buildings, TV antennae, towers or any other tall objects which are not readily measurable by direct measurement. The measurements given are suggested. It is a hypsometer (hypso-height, metron-measure).

Project 5

Purpose

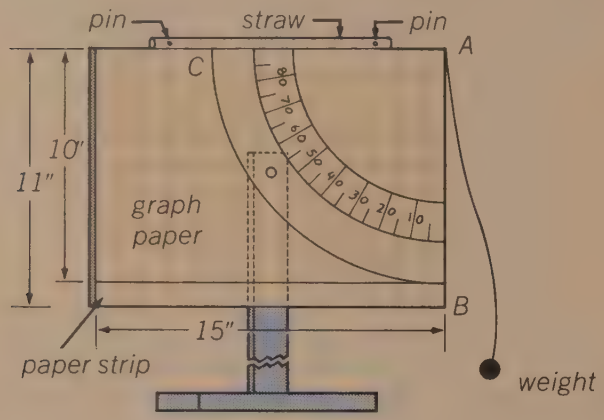
To construct and learn how to use a hypsometer.

Method

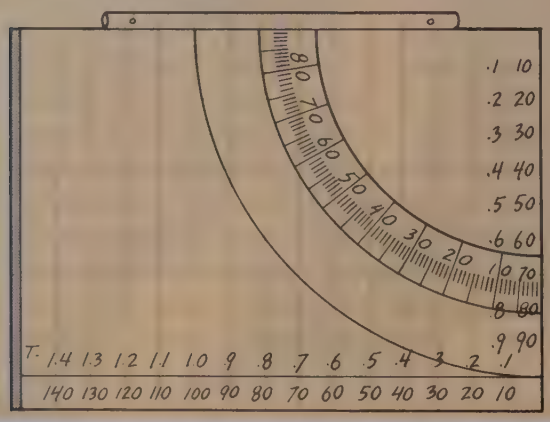
1. Obtain
   - 1 heavy cardboard or light plywood 11″ × 15″
   - 2 sheets of graph paper, 10″ × 10″, 10 squares to the inch
   - 1 quadrant of polar coordinate graph paper
   - 1 plastic soda straw
   - 2 fine straight pins
   - string, glue
   - weight (heavy eraser, fish line sinker, etc.)
   - 1 stand base and rod with bolt and wing nut

Here a student is using a hypsometer to find the height of a tree.

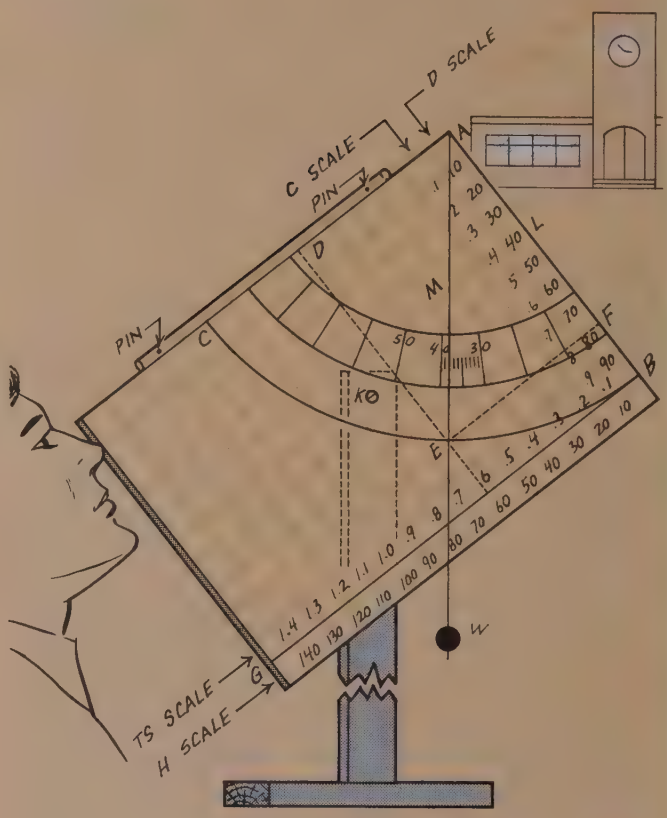
2. Glue the 2 sheets of graph paper on the plywood. On the bottom glue a plain white strip 1 inch wide as shown.

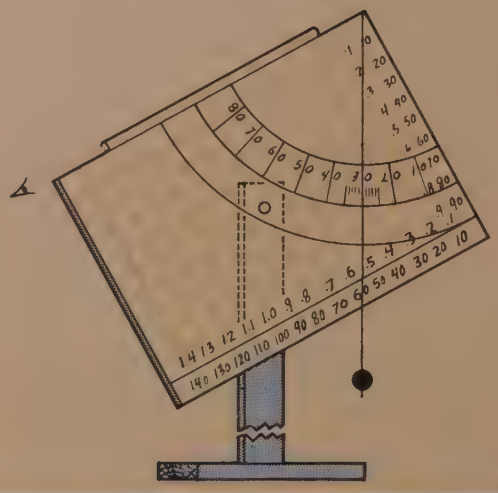


3. Trim the polar coordinate paper as shown and glue in place.
4. Attach string and weight as shown.
5. With centre *A* and radius 10 inches draw arc *BC*.
6. Glue the straw to the top of the instrument Exercise care to get proper alignment with edge of board.
7. Mount the plywood to the stand at point *K* by means of the winged nut and bolt.
8. Put a pin through each end of the straw to act as the horizontal half of a set of cross hairs ⊖. Snip to length and cement in place.
9. The instrument is now ready to use. Study the diagram closely and label the scale on your instrument as shown.



10. Sight through the straw lining up the pins and the top of the building.





11. The hypsometer may be reversed to measure the angle of depression.

12. Let us check our instrument for accuracy. Tilt the hypsometer for the following angles of inclination. Fill in this chart.

FUNCTIONS OF ANGLE	HYPSOMETER READING	TABLES	ERROR
(a) sine 50°	0.77	0.766	0.004
(b) cosine 30°			
(c) tan 15°			
(d) tan 65°			
(e) cosine 48°			
(f) sine 72°			

Project 6

Purpose

To find the height of your school building.

Method

1. Choose a point where the hypsometer base will be on the same level as the building.
2. Set up the hypsometer and take a sighting of the top of the school building.
3. Determine by measuring, the distance from the hypsometer to the building. Remember the distance from the hypsometer to the building must be to that part of the building directly below the high point you have sighted.
4. Obtain data from 3 trials. Use a different position for the hypsometer each time.

5. Record the data and complete this table:

TRIAL	HEIGHT OF HYPSOMETER	HEIGHT ABOVE HYPSOMETER	TOTAL HEIGHT
1			
2			
3			

Average height: _____

6. If your building has two or more floors use the hypsometer to find the measure of the distance from the ground to an upper window. Repeat twice and average.
7. Mark a spot on the ground some distance (measure this distance) from the building. Go to the window (in part 6) and use the hypsometer to find the height of the window. Repeat using two different spots on the ground. Average the results.
8. Compare the average of the heights obtained in parts 6 and 7 with the actual height obtained by using a long tape measure (100 feet).
9. What is the percentage error in part 8?
10. Find the height of a building in your vicinity.

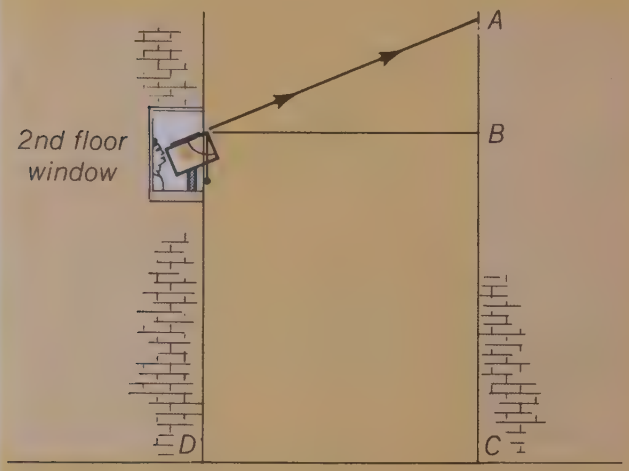
Project 7

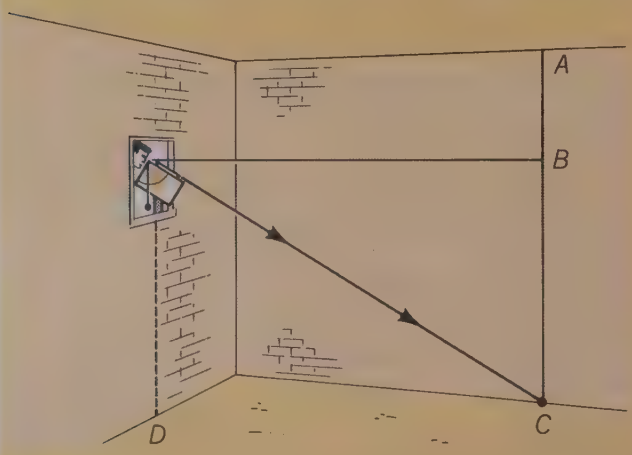
Purpose

To find the height of an object from a point that is on a level higher than the base of the object.

Method

1. Choose a second floor window that faces a wall of another building, either directly opposite or obliquely opposite.





2. Find the distance from the window to the wall opposite ($\overline{CD}$).
3. Use the hypsometer to find the total height of the wall in two steps.
   (a) measure $\overline{AB}$.
   (b) measure $\overline{BC}$.
   (c) total measure (sum of (a) and (b)).

This student is using a stadia tube to determine distance.

Project 8

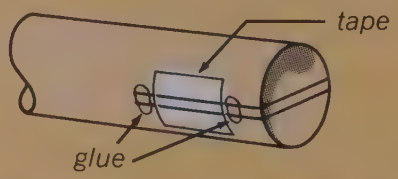
Purpose

To make, understand, and use a stadia.

Method

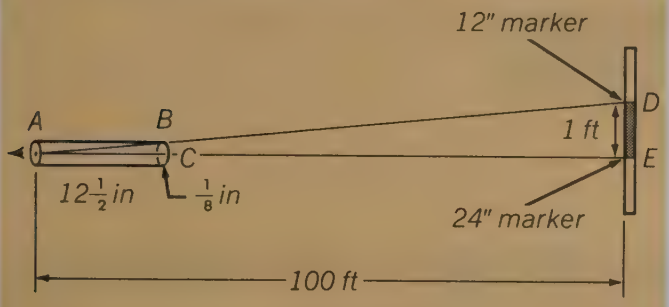
1. Obtain a tube with a diameter of about 1" ($\frac{1}{2}$" to $1\frac{1}{2}$") *exactly $12\frac{1}{2}$ inches in length.*

2. On one end glue an endpiece.
3. Put a pinhole in the centre of this endpiece.



4. Place two *fine* threads across the open end near the diameter, parallel to each other exactly $\frac{1}{8}$ inch apart.
5. Fasten firmly with threads taut.

6. The instrument is now ready to use. Place this instrument to your eye (pinhole at your eye) and sight at a yardstick 100 feet from you. Between the cross-threads you should see one foot of yardstick. (You may have to place bright markers on the yardstick at the 12-inch and 24-inch points in order to see at this distance.)



If you see more than 1 foot, either the tube is less than $12\frac{1}{2}$ inches in length or the cross-threads are more than $\frac{1}{8}$ inch apart. Make the correction until at 100 feet you see 1 foot of the yardstick between the crossthreads.

Here is why the instrument works as it does. Consider the similar triangles as labelled.

$\triangle ABC$ is similar to $\triangle ADE$

(1) $\frac{m(\overline{AF})}{m(\overline{BC})} = \frac{m(\overline{AG})}{m(\overline{DE})}$ (Are altitudes of similar triangles in the same ratio as the sides?)

$\frac{m(\overline{AF})}{m(\overline{BC})} = \frac{12\frac{1}{2}''}{\frac{1}{8}''}$ replacements

(2) $\frac{m(\overline{AF})}{m(\overline{BC})} = \frac{100}{1}$ number facts

(3) $\therefore \frac{m(\overline{AG})}{m(\overline{DE})} = \frac{100}{1}$ transitivity property from (1) and (2)

$\therefore$ if $m(\overline{AG})$ is 100 feet, then $m(\overline{DE})$ is 1 foot.

We may now make two conclusions:
(a) $m(\overline{DE})$ is $\frac{1}{100}$ of $m(\overline{AG})$.
(b) $m(\overline{AG})$ is 100 times $m(\overline{DE})$.

7. Use the stadia and a yardstick to find the dimensions of your school grounds. (Attach different colored bright tapes or paper at every inch of the yardstick in order that it can be more easily seen from a distance.

8. Find the outside dimensions of your school.

10·7 Extra Activities

1. Obtain a transit and learn how to use it.

2. If your school is using P.S.S.C. Physics (Physical Science Study Committee Physics) obtain from the science teacher the instruments used to measure distances. There are two instruments, one for measuring large distances and one for measuring small distances. Learn to use both.

3. How does radar work? What is sonar?

Air traffic controller monitors Precision Approach Radar.

Using the Library

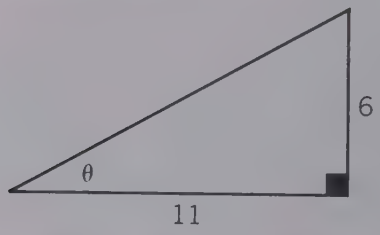
A sundial can be used to tell time. Refer to your library to find instructions on how to make one. Then construct a sundial that is suited to your latitude.

REVIEW EXERCISE 10

1. Use tables to find cos 63°.

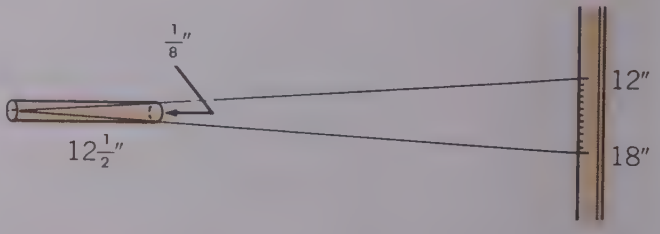
2. Use tables to find θ when
   (a) $\cos\theta° = 0.8070$ (b) $\tan\theta° = 0.7500$

3. Find θ given the triangle shown.



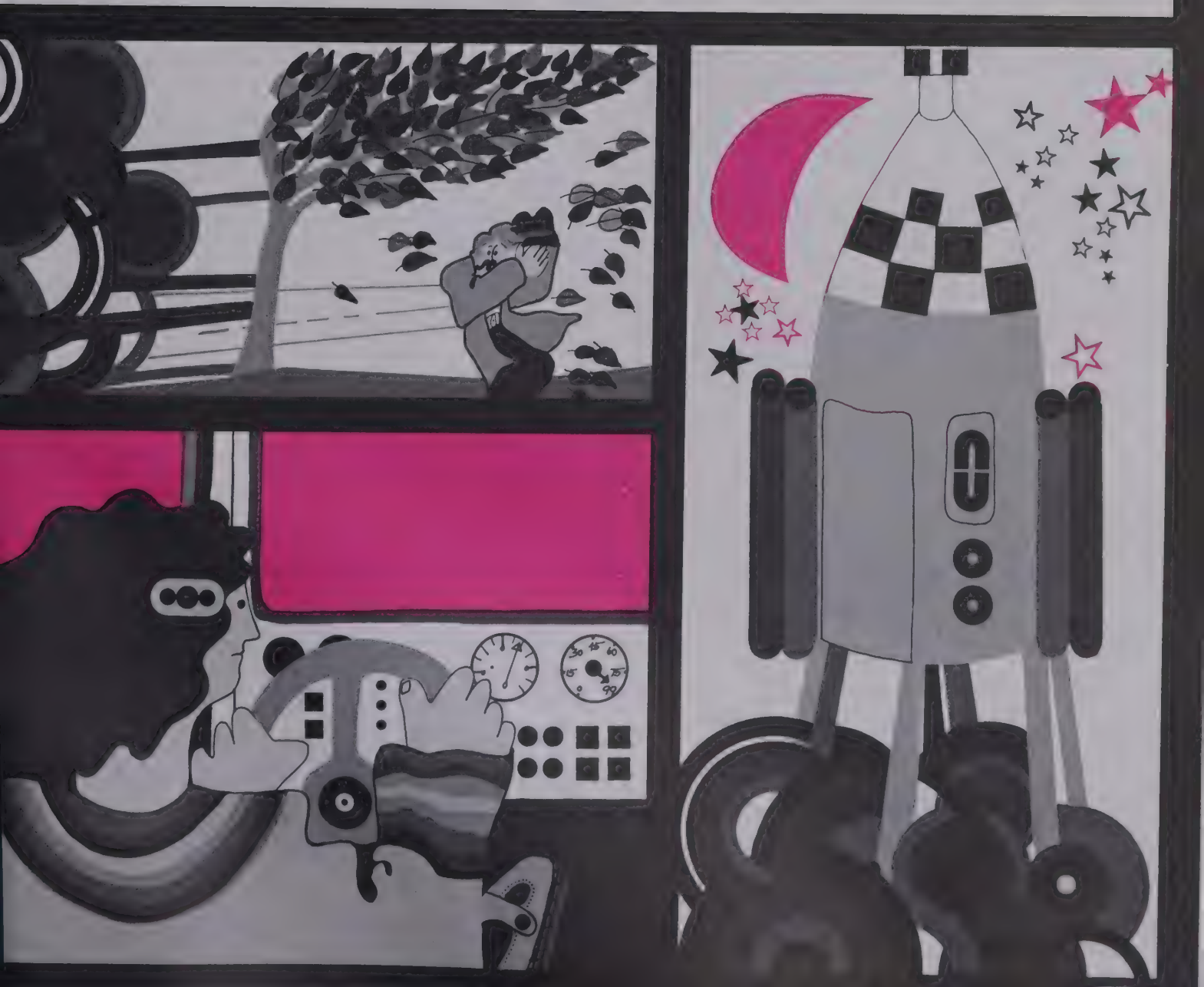
4. The hypsometer showed a tangent value of 0.7000. What is the distance from a tree to the hypsometer if the instrument is 10 feet above the base of the tree?

5. The stadia was used and the data recorded as shown in the diagram. How far is the stadia from the yardstick?



6. Use the hypsometer to find the height of the wall in your classroom.

Chapter Eleven
VECTORS



11·1 Introduction

Read the following six statements and note the differences between the quantities or magnitudes in the first three statements and those in the last three statements.

John is *17 years* old.

Apollo 8 carried *3 astronauts.*

The volume of the cylinder is *8π cubic feet.*

The jet flew *400 miles northwest.*

The helicopter lifted *upward* with a *10 ton force.*

The wind blew with a velocity of *60 miles per hour from the northwest.*

In the first three examples the underlined portion refers to "how many units". The number is the magnitude. The examples tell how old, how many astronauts and how many cubic feet.

> Quantities that tell how many units only are called scalar quantities.

The scalar quantities are to be compared with the quantities in the last three statements. These three quantities tell how many miles northwest, how many tons force upward, and how many miles per hour from the northeast.

> Quantities that tell direction as well as how many units are called vector quantities.

Segments which indicate direction are called directed segments. Compare the segments in Figure 11-1 with those in Figure 11-2. The length of the segment indicates its magnitude and the attitude of the segment along with the arrowhead indicates the direction.

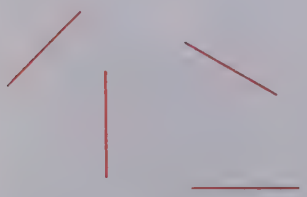


Figure 11-1

Figure 11-2

> Directed segments which have the same direction and magnitude are said to be equivalent directed segments.

Discuss which of the directed segments in Figure 11-3 are equivalent.

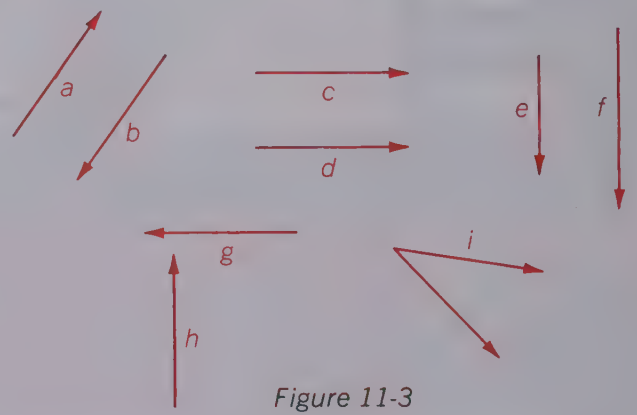


Figure 11-3

On the Cartesian plane draw a directed segment starting at the origin and ending at (4,0). We place an arrowhead on the terminal point of the segment to show that the direction is from left to right.

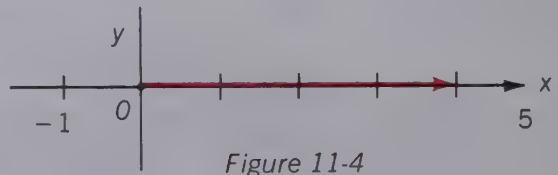


Figure 11-4

On another Cartesian plane draw a directed segment starting at (2,0) and going to (−3,0). Again, we place an arrowhead on the terminal point of the segment to indicate that the direction is from right to left.

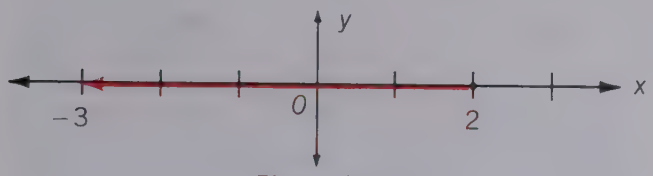


Figure 11-5

These two examples might illustrate a walk 4 blocks (or miles) due east (east is usually represented as a movement to the right) and 5 blocks (or miles) due west respectively. On the other hand, the examples might illustrate a 4-pound force to the right and a 5-pound force to the left respectively.

On a Cartesian plane draw a segment with the initial point at the origin and the terminal point at (3,4). Place the arrowhead appropriately. This directed segment might represent any number of things. It might represent the single pass of a hockey puck in which the forward displacement (to the right) is 3 units and the lateral displacement is 4 units. (See Section 10-3, Project 3). The puck actually moved 5 units.

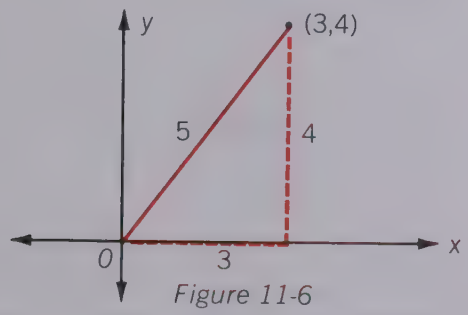
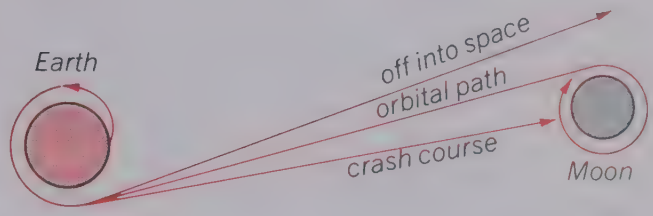


Figure 11-6

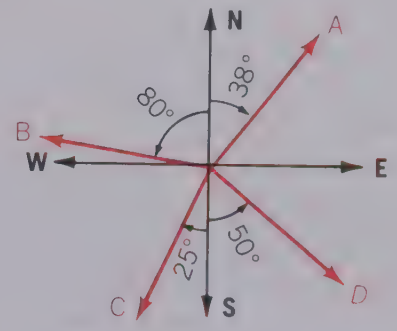
Notice that in these examples we have indicated the amount of movement and the direction of the movement. In the first example we can use (4,0) to indicate that the movement was 4 units to the east and no units north or south. In the second example we can use (−5,0) to indicate that the displacement was 5 units to the west and no units north or south. In the third example we can use (3, 4) to indicate that the displacement to the right is 3 and the displacement upward is 4. The ordered pair tells us magnitude and direction (sometimes indirectly as in the third example).

The directed segments we have shown are geometric representations of vector quantities. We have indicated in each case magnitude and direction. It is not sufficient to describe a displacement by magnitude alone. The direction of Apollo 8 on its moon flight was very important, for if the space capsule had been one degree off course it would have either crashed into the moon or missed its possible orbital path altogether and gone off into space.



Similarly, to specify force, both magnitude and direction need to be stated. We realize the direction of the force is very important if we consider a car stuck in a snowbank. Should we push forward on the car or sit on the car?

To be defined, vector quantities require both magnitude and direction. Since directions are so important we should know how to speak about them. Directions are often given as the measure of the acute angle with the North-South line.



Vector $\overrightarrow{OA}$ is 38° east of north (N38°E which is read *north, 38° east).*

Vector $\overrightarrow{OB}$ is 80° west of north (N80°W).

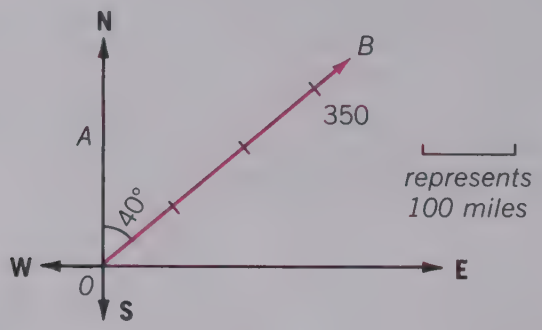
Vector $\overrightarrow{OC}$ is 25° west of south (S25°W).

Vector $\overrightarrow{OD}$ is 50° east of south (S50°E).

Example 1

Draw to scale a vector to represent a car trip, of 350 miles north 40° east.

Solution



$m \angle AOB = 40°$

$m(\overline{OB}) = 350$ units

Draw the figure to scale.

Vector $\overrightarrow{OB}$ is the required vector.

EXERCISE 11·1

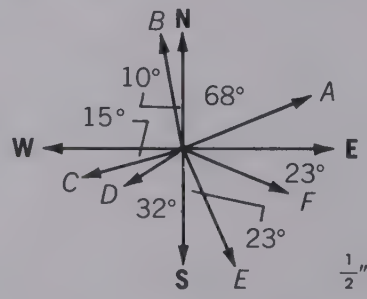
1. Identify each as a scalar quantity or a vector quantity.
   (a) Number of snowmobiles in the club.
   (b) Position of Toronto relative to North Bay.
   (c) Your age.
   (d) The path of the ferry, William Carson, from Sidney, Nova Scotia, to Port Au Basque, Newfoundland.
   (e) The altitude of Mt. Edith Cavell.
   (f) The distance Toronto is from the 49th parallel.
   (g) The force of a towtruck on a stalled car.
   (h) The path of an airplane in flight.
   (i) The movement of a skier on a slope.
   (j) The volume of a rectangular prism.

2. On a Cartesian plane draw these vectors. Use the origin as the initial point of each.
   (a) (5,4) (b) (−3,6) (c) (0,4)
   (d) (−4,−4) (e) (5,−2) (f) (−3,0)

Stresses and strains in a ski chair lift require that the equipment be of high standard. Technologists and engineers must ensure that the equipment will stand the stresses.

3. On a Cartesian plane draw these vectors. Use (2,3) as the intitial point of each.
   (a) (4,3) (b) (5,6) (c) (−4,−3)
   (d) (6,0) (e) (−2,−3) (f) (4,−1)

4. Name the direction of each vector. What is the magnitude of each?

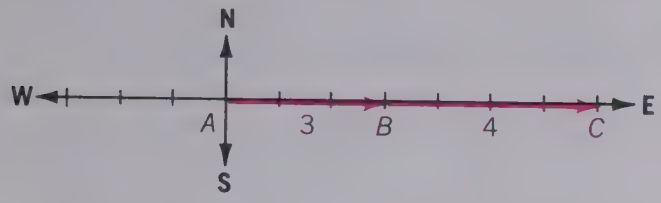


$\frac{1}{2}$" = 10 units

5. On a Cartesian plane draw a vector to represent each of the following.
   (a) A plane flies 600 miles due east.
   (b) A plane flies 475 miles southeast (S45°E).
   (c) a force of 80 pounds upward 70° to the horizontal (the positive *x*-axis);
   (d) a force of 45 pounds vertically downward.
   (e) a velocity of 450 miles per hour 70° east of north (N70°E);
   (f) a velocity of 375 miles per hour S60°W.

11·2 Addition of Vectors

If a boy walks three blocks east, stops, then walks four more blocks east we know he has walked 7 blocks east and that he is 7 blocks from where he started. Let us represent this on the Cartesian plane.



We labelled the initial point of the first vector A, and since it is the starting point it is customary to start at the origin. The terminal point of the first vector is (3,0). Call this point B.

> We use the symbol $\overrightarrow{AB}$ for the vector whose initial point is A and whose terminal point is B.

You should note that the symbol for a vector is the same as that for a ray: $\overrightarrow{AB}$. In this chapter, if we wish to discuss a ray, we write "ray AB" and the symbol $\overrightarrow{AB}$ will be used exclusively for "vector AB".

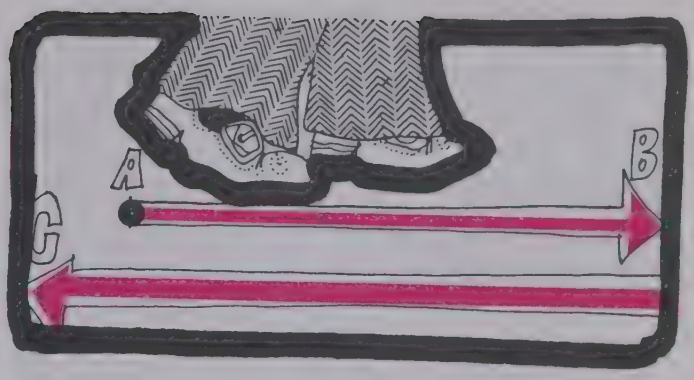
The second vector is $\overrightarrow{BC}$. The sum of the two vectors is written

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

> The sum of two vectors is the one vector that has the same effect as the two vectors.

In the example above, we ask, "What one walk accomplishes as much as the two walks?" Obviously, a walk of 7 blocks east.

Similarly, a walk 3 blocks east, followed by a walk 4 blocks west places the boy one block west of where he started. If we follow the custom and interpret motion eastward as positive, then motion westward is negative. See the following diagram.



$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

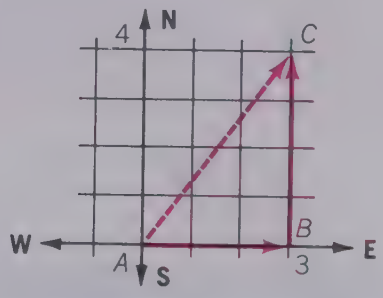
We can see that a walk from A to C puts the boy in the same position as the two walks from A to B and then from B to C.

Example 1

Brian leaves his house and walks 3 blocks east then 4 blocks north. What single walk (if there are no obstacles) could Brian take from his home that would bring him to the same spot?

Solution

1. Draw the diagram to scale on a Cartesian plane as shown.



2. From the diagram we know that $3 + 4 \neq 7$ in the terms of vector addition. (We know that $m(\overline{AC})=5$). According to the diagram the walk from A to C puts Brian at the same point as the walk from A to B and then from B to C.

   In vector notation we write

   $$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

   3 blocks (E) + 4 blocks (N) = 5 blocks (N37°E).

   The direction is found by measurement. A single walk of five blocks from home at N37°E will bring Brian to the same point as the two walks did.

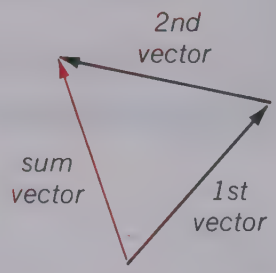
This is what we shall mean by vector addition:

> The sum of two vectors is that one vector which has the same effect as the two vectors.

To add vectors geometrically:

START

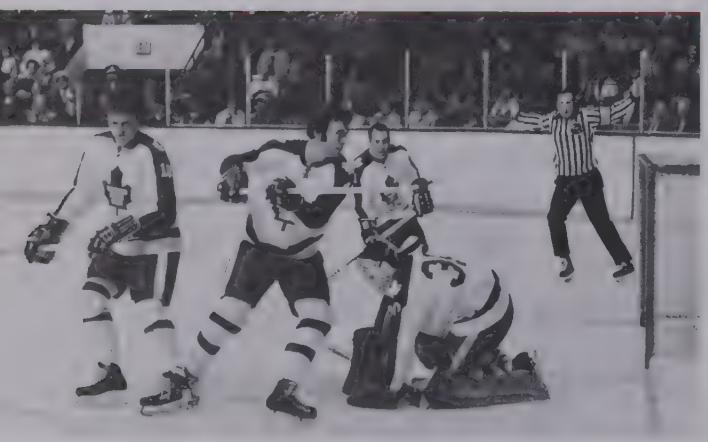
DRAW FIRST VECTOR TO SCALE IN PROPER PROPORTION



DRAW SECOND VECTOR WITH INITIAL POINT AT TERMINAL OF FIRST VECTOR AND WITH PROPER DIRECTION

DRAW **SUM** VECTOR WITH INITIAL POINT THE SAME AS INITIAL POINT OF FIRST VECTOR AND TERMINAL POINT THE TERMINAL POINT OF SECOND VECTOR. DIRECTION IS INDICATED.

STOP

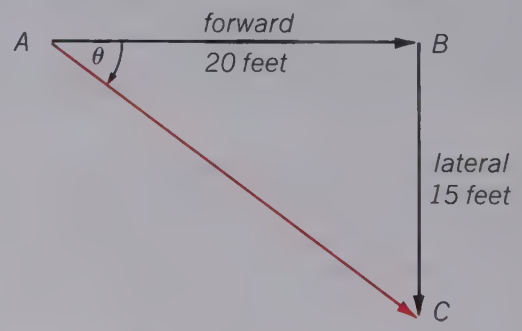


Example 2

A hockey player, A, standing in the face-off circle in his own end, passes the puck directly foward 20 feet to another player, B, who, in turn, passes it 15 feet directly across the ice to a third player, C. What was the displacement of the puck and in what direction?

Solution

In terms of vectors: magnitude of $\overrightarrow{AB}$ is 20 feet, forward; magnitude of $\overrightarrow{BC}$ is 15 feet, lateral.



Displacement (the sum) is $\overrightarrow{AC}$.

$$m(\overline{AC}) = \sqrt{20^2 + 15^2}$$
$$= 25.$$

Direction by measurement:

$$m\angle BAC = 37°$$

Therefore, the puck was displaced 25 feet at an angle of 37° from the forward line.

Optional Solution

To find θ:

$$\tan\theta = \frac{m(\overline{BC})}{m(\overline{AB})}$$
$$\tan\theta = \tfrac{15}{20}$$
$$\therefore \quad \theta = 37°$$

To find $m(\overline{AC})$:

$$\sin\theta = \frac{m(\overline{BC})}{m(\overline{AC})}$$
$$m(\overline{AC}) = \frac{m(\overline{BC})}{\sin\theta}$$
$$m(\overline{AC}) = \frac{15}{\sin 37°}$$
$$m(\overline{AC}) = \tfrac{15}{0.60}$$
$$m(\overline{AC}) = 25$$

Therefore, the sum of the two displacements is 25 feet 37° from the forward line.

EXERCISE 11·2

1. Draw vectors on the Cartesian plane to represent the addition of the vectors in each case. Find the magnitude and direction of the sum. (If you took the unit on trigonometry use trigonometric solution where possible)
   (a) A car trip of 4 miles east followed by another trip 7 miles east.
   (b) A motorcycle ride 6 miles north followed by a ride 7 miles south.
   (c) A force of 100 pounds acts directly upward on a weight of 90 pounds.
   (d) An airplane with a velocity of 200 miles per hour in still air flies north into a wind from the north which has a velocity of 60 miles per hour.
   (e) An airplane ride 600 miles north followed by an airplane ride 400 miles east.
   (f) A force of 70 pounds acting due north and a force of 80 pounds acting due east work at the same time on a body.

2. A basketball player dribbles straight down the floor for $2\frac{1}{2}$ seconds at 22 feet per second. He then passes the ball directly across the floor for $\frac{1}{4}$ second at 80 feet per second. Draw a vector diagram to show the sum of the two vectors.

3. A ship sails N38°W. Its speed is 20 knots. Another ship sails from the same position at the same time. It travels N52°E at 35 knots. How far apart are the two ships after two hours? (A knot is equivalent to one nautical mile per hour.)

11·3 Navigation

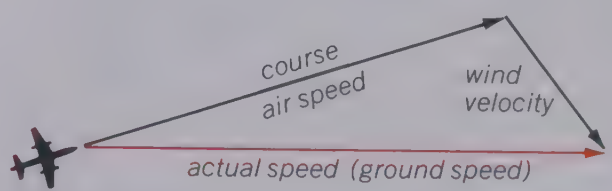
Vectors are used to solve problems in navigation. We must be aware of the words which occur in these problems. We will refer to an airplane. Similar terms are used for ships at sea or in rivers.

The *actual speed* (or *ground speed*) is the speed relative to the ground over which it is flying.

The *speed in still air* (or *air speed*) is the speed the plane flies in still air. It does not take into account the wind speed which alters the air speed to give the ground speed.

The *track* is the actual path of the plane.

The *course* is the path planned. This is the actual path of the plane provided there is no wind or other influence. A wind may blow the plane off course.



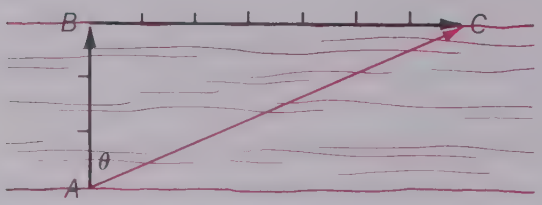
In navigation, the navigator must make allowances for wind (or current). A wind could blow a plane off course. A person in a rowboat must actually point his boat slightly upstream in order to arrive at a point on the opposite bank directly across from his starting point.

Example 1

A swimmer attempts to swim across a river. His speed in still water is 3 miles per hour. The river flows at 7 miles per hour. What is the speed of the swimmer relative to the shore? What is the direction of his movement?

Solution

Draw the diagram to scale.



We have two vectors given:

(i) The swimmer's velocity and direction: $\vec{AB}$ has a magnitude of 3 and a direction of "across" the river.

(ii) The velocity and direction of the river current: $\vec{BC}$ has a magnitude of 7 and a direction of "down" the river.

The third vector which we must find is $\vec{AC}$. It is the sum of $\vec{AB}$ and $\vec{BC}$. The magnitude of $\vec{AC}$ is

$$\sqrt{3^2 + 7^2} \text{ or } \sqrt{58}$$

By measurement, $m\angle BAC$ is 67°.

The speed of the swimmer relative to the river bank is approximately 7.6 miles per hour. The direction is 67° downstream from the "across" line.

Optional solution

To find $m\angle BAC$:

$$\tan\theta = \frac{7}{3}$$

$$\therefore \quad \theta \simeq 67°$$

To find $m(\overline{AC})$:

$$\sin 67° = \frac{7}{m(\overline{AC})}$$

$$m(\overline{AC}) \simeq \frac{7}{0.92}$$

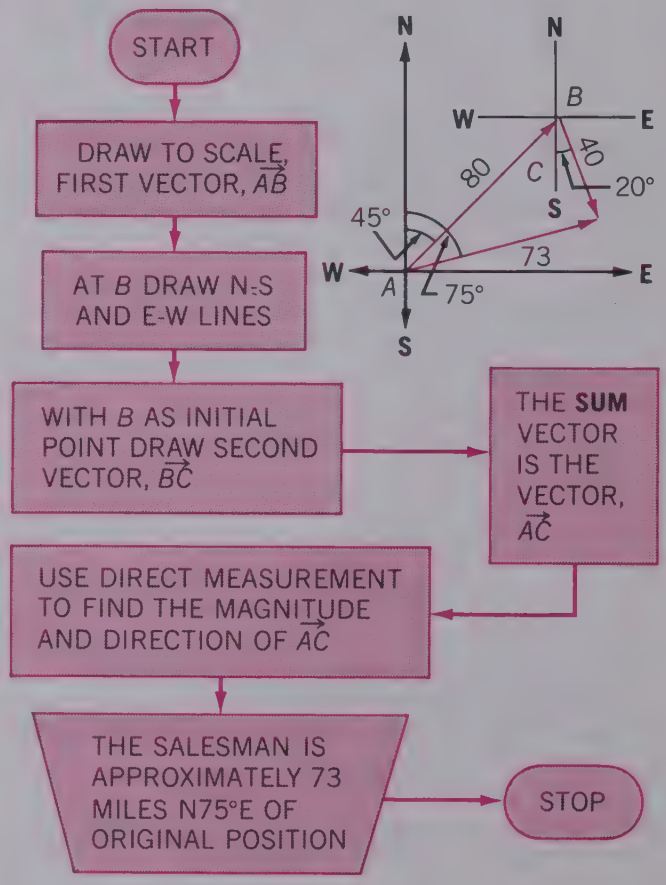
$$m(\overline{AC}) \simeq 7.6$$

Up to this point we have worked problems in which the two vectors are at right angles to each other. We shall now look at examples in which the vectors are not at right angles.

Example 2

A salesman drives his car 80 miles northeast (N45°E). He then turns and drives south 20° east for 40 miles. How far is the salesman and in what direction from his original position?

Solution

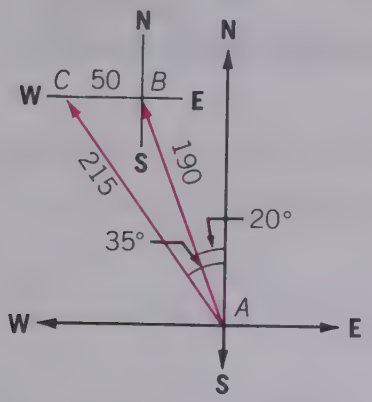


Example 3

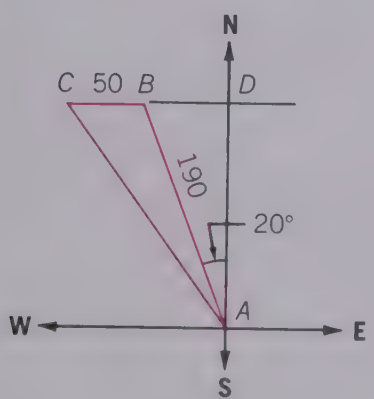
An airplane flies N20°W at 190 miles per hour air speed. A wind is blowing at 50 miles per hour from the east. Find its ground speed and its track.

Solution

1. Draw to scale the first vector ($\vec{AB}$).



2. At the terminal end of the first vector draw another set of N-S and E-W lines.
3. With B as the initial point draw the second vector to scale ($\vec{BC}$).
4. Draw in the sum vector ($\vec{AC}$).
5. By measuring determine the magnitude of $\vec{AC}$ and the direction.
6. The ground speed of the airplane is 215 miles per hour and the track is N35°W.
7. If you studied the chapter on trigonometry, find the magnitude and direction of vector $\vec{AC}$ by trigonometry.



(a) Use sine relation in $\triangle ABD$ to find $m(\overline{DB})$ first.

$$\sin 20° = \frac{m(\overline{DB})}{190}$$

$$m(\overline{DB}) = \blacksquare$$

(b) Now use cosine relation in $\triangle ADB$ to find $m(\overline{AD})$.

$$\cos 20° = \frac{m(\overline{AD})}{190}$$

$$m(\overline{AD}) = \blacksquare$$

(c) Use tangent relation in $\triangle ADC$ to find $m\angle DAC$

$$\tan \angle DAC = \frac{m(\overline{DC})}{m(\overline{AD})}$$

$$m\angle DAC = \blacksquare$$

(d) Use sine relation in $\triangle ADC$ to find $m(\overline{AC})$.

$$\sin m\angle DAC = \frac{m(\overline{DC})}{m(\overline{AC})}$$

$$m(\overline{AC}) = \blacksquare$$

The ground speed is ■ mph. Its track is N■°W.

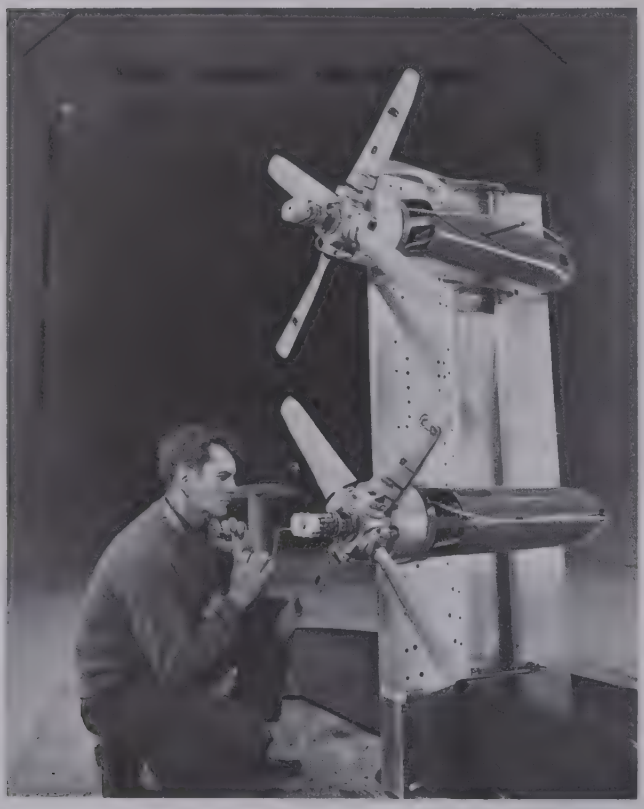
EXERCISE 11·3

1. A rowboat sets a course directly across a river which flows at 15 miles per hour. The water speed of the boat is 30 miles per hour. What is the actual path of the boat (direction and speed)?

2. An airplane has an air speed of 150 miles per hour. Its course is S38°W. A wind of 40 miles per hour blows from the east. What is the ground speed and track of the plane?

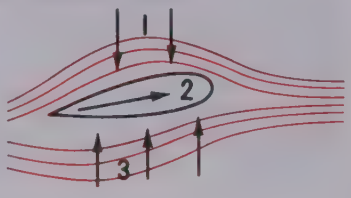
3. A jet fighter with air speed of 800 miles per hour plans a course N52°E. It encounters a jet stream (the wind at high altitudes) blowing from N23°E at 110 miles per hour. What is the track and ground speed of the jet fighter?

4. A ship entering harbour with water speed of 8 knots proceeds on course N33°E. The tide is running from N23°W at 6 knots. What is the ship's speed relative to the harbour? What is the direction of its track?



This is a model of a short take-off and landing aircraft being tested in NRC's 30-foot wind tunnel. An airplane wing is subjected to three forces:

1. *its weight pulling downward,*
2. *motor force pushing it ahead,*
3. *wind pressure holding it up.*



11·4 Scalar Multiplication of Vectors

If you were asked to find the value of
67 + 67 + 67 + 67 + 67 + 67 + 67 + 67 + 67 + 67,
how would you do it? Would you add? Not likely! Multiplication in the set of real numbers produces the same result as successive addition. Let us see if there is something similar to this in vectors.

The driver of a snowmobile drives N70°E for 4 miles. He stops, then proceeds in the same direction for another 4 miles. He stops again then continues in the same direction for 4 more miles. In what direction and how far is he from where he started?

From Figure 11-7 we see that he is 12 miles N70°E of where he started.

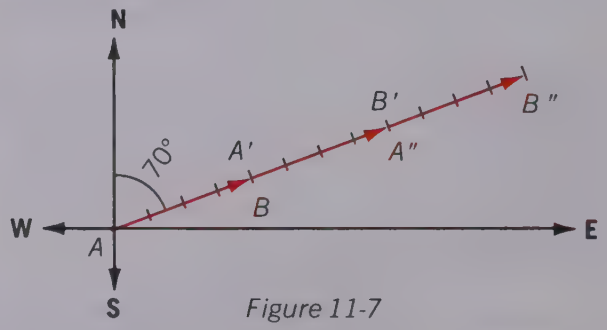


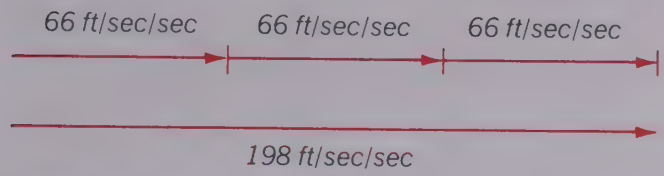
Figure 11-7

The sum is (4 miles N70°E) + (4 miles N70°E) + (4 miles N70°E) = 12 miles N70°E. It seems we could have multiplied the magnitude of the vector $\overrightarrow{AB}$ by the scalar 3. By this we mean we have 3 units where each unit is a vector with magnitude 4 and direction N70°E. Hence, we mean

3(4 miles N70°E) = 12 miles N70°E.

Let us look at another situation.

A spacecraft has 3 rockets of equal power. Each rocket can accelerate the spacecraft forward at the rate of 66 feet per second per second (66 ft/sec/sec). What is the sum of the effect of the 3 rockets? We can show this by a vector diagram:



or we may write:

66 ft/sec/sec forward + 66 ft/sec/sec forward
+ 66 ft/sec/sec forward = 198 ft/sec/sec forward.

We think of scalar multiplication of vectors in this case as 3(vector) in which the vector is 66 ft/sec/sec forward.

Hence,
3(66 ft/sec/sec forward) = 198 ft/sec/sec forward

We have now established what we mean by scalar multiplication of vectors. Note, this is *not* multiplication of vectors by vectors.

EXERCISE 11·4

1. Find the product in each of the following scalar multiplications of vectors:
   (a) 6(90 miles per hour N40°E).
   (b) 12(78 feet S34°E).
   (c) 36(54 ft/sec/sec N).
   (d) 5(80 pounds force straight ahead).
   (e) 28(63 pounds force downward).

2. Draw vector diagrams to illustrate:
   (a) 2(4 ft. N30°W) + 2(3 ft. S40°W)
   (b) 2(4 ft. N30°W + 3 ft. S40°W).
   Do you think scalar multiplication is distributive over vector addition?

3. Draw vector diagrams to illustrate:
   (a) 3(2 lbs. N) + 3(4 lbs. E).
   (b) 3(2 lbs. N + 4 lbs. E).
   What property does this illustrate?

11·5 Sum of Three or More Vectors

A taxi driver leaves the bus station and drives east 4 blocks on 8th Avenue, then turns south for 5 blocks on 6th Street. He then turns east on 13th Avenue for 6 blocks and then turns south on 12th Street for 3 blocks. How far is he from the bus station (as the crow flies) and in what direction?

Let us show this by a vector diagram.

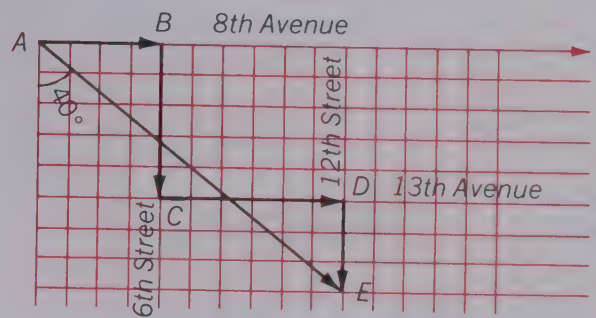


Figure 11-8

We follow the rule for addition: Join the initial point of his trip and the terminal point of his trip with a directed segment. By measurement the magnitude of the vector is $12\frac{3}{4}$ blocks. The direction is S40°E.

If there are no blocked streets or avenues, we recognize that the taxi driver could have gone east on 8th Avenue for 10 blocks then south on 12th Street for 8 blocks.

In terms of vectors it seems we have moved $\vec{CD}$ so that the initial point of $\vec{CD}$ is the terminal point of $\vec{AB}$. Similarly, we can consider BC to be moved as shown in Figure 11-9. We then add vectors AB and DE. In other words, it seems

$$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} = \vec{AB} + \vec{CD} + \vec{BC} + \vec{DE}$$

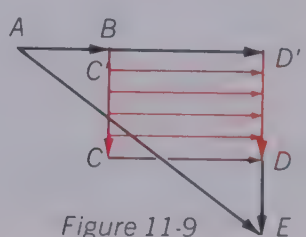


Figure 11-9

EXERCISE 11·5

Use vector diagrams to solve these problems.

1. A delivery boy leaves a store to make deliveries. On his motorbike he goes east 3 blocks, south 6 blocks, east 4 blocks, and north 2 blocks. How far (as the crow flies) is he from the store and in what direction?

2. A plane leaves the base and flies due north for 350 miles. It then experiences difficulty. In turns and flies S60°E for 125 miles. Because of heavy head winds it turns again and flies S10°W for 250 miles. How far from the base is the plane? In what direction must the plane fly now to reach base?

3. A sailboat is heading in a northerly direction into a wind. It is zigzagging with these distances and directions:

   (a) N35°E for $\frac{1}{2}$ mile
   (b) N50°W for $\frac{3}{4}$ mile
   (c) N40°E for 1 mile
   (d) N45°W for 2 miles.

   Find the sum of the 4 vectors (magnitude and direction).

Projects

Project 1

Purpose

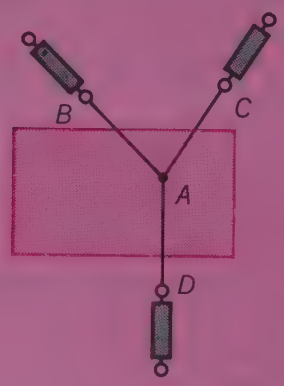
To study the relation of two vectors and their sum.

Background

If two equal forces act in opposite directions their sum is zero. We say the forces are in equilibrium. If the two forces are represented by $\vec{BA}$ and $\vec{BC}$ the magnitude of $\vec{BA}$ is equal to the magnitude of $\vec{BC}$ but the two vectors act in opposite directions.

Method

1. Obtain 3 spring balances.
2. Knot together three strings each about 1 foot long.
3. Attach a spring balance to the other end of each string. Arrange as shown.



4. Place a large sheet of paper underneath. Place a dot on the paper directly under the knot at *A*.
5. Each of three students pulls on a string balance but keep the knot directly over point *A*.
6. When the system is steady, each student takes the reading on his balance, while a fourth student marks accurately the direction of each string.
7. Remove the apparatus; on the paper draw the three vectors to scale.
8. Use $\vec{AB}$ and $\vec{AC}$ as two adjacent sides of a parallelogram.

9. Complete accurately the parallelogram.
10. Draw the diagonal of the parallelogram that contains point *A*.
11. Find the magnitude of the diagonal.
12. Is the diagonal equal in magnitude to the third vector $\vec{AD}$?
13. Is the diagonal vector contained in the same line as the third vector $\vec{AD}$?
14. Is *AD* equal in magnitude to the sum of $\vec{AB}$ and $\vec{AC}$?
15. Repeat twice steps 4 through 14 above for various positions of the spring balances.
16. What is your conclusion?

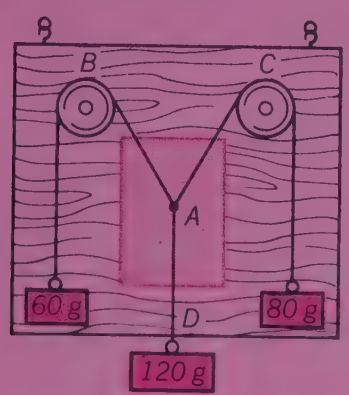
Project 2

Purpose

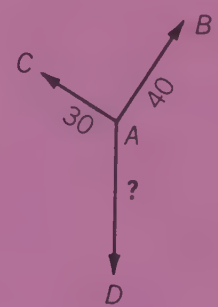
To explore the relationship between forces in equilibrium.

Method

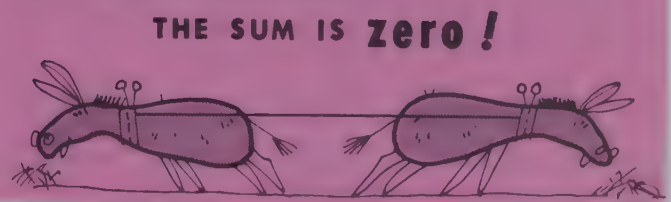
1. Obtain 2 single pulley wheels and attach them by means of nails or smooth shank screws to a piece of plyboard about 16″ × 16″. Hang the board vertically by eye hooks.



2. Knot together 3 pieces of stout cord each about 2 feet long. On the end of each cord attach a weight and arrange the cords and weights over the pulleys as shown, such that the system is in equilibrium.
3. Record the magnitude of the weights.
4. Place a sheet of paper on the board behind the apparatus.
5. Mark accurately the position of the knot *A* and the direction of each cord.
6. Remove the paper and draw each force vector to scale.
7. Complete the parallelogram in which $\vec{AB}$ and $\vec{AC}$ are adjacent sides.
8. Draw the diagonal of the parallelogram that contains point *A*.
9. Find the magnitude of the diagonal.
10. Is the diagonal equal in magnitude to the third vector $\vec{AD}$?
11. Is the diagonal vector contained in the same line as the third vector?
12. Is the third vector equal in magnitude to the sum of the other two vectors?
13. Is the diagonal vector equal in magnitude to the sum of the two vectors that form adjacent sides of the parallelogram?
14. Repeat twice steps 2 through 13 above for various weight combinations.
15. The three forces represented by the vector below are in equilibrium. What is the magnitude of the force represented by $\vec{AD}$ if the magnitudes of $\vec{AC}$ and $\vec{AB}$ are 30 grams and 40 grams respectively?



16. What is your conclusion?



11·6 Equivalent Vectors

Earlier we said that vectors (directed segments) with the same direction and the same magnitude are equivalent vectors. In the problem in the previous section (Figure 11-9) $\overrightarrow{CD}$ is equivalent to $\overrightarrow{C'D'}$.

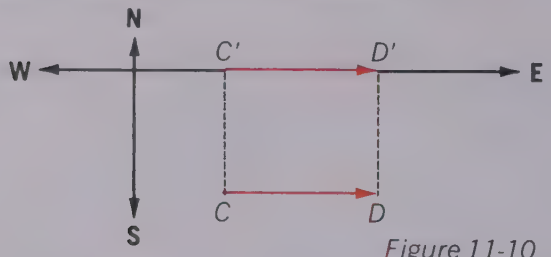


Figure 11-10

You will recall that in the set of real numbers we may replace a number with an equivalent number. *In vectors we may replace one vector with an equivalent vector.* We use this principle in certain problems.

Consider a car being towed by two ropes as shown.

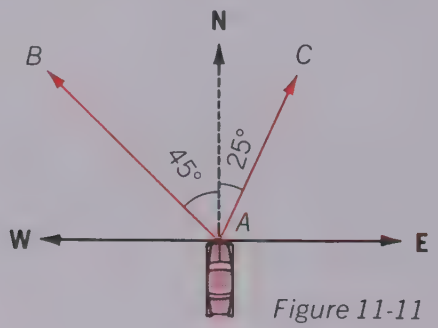


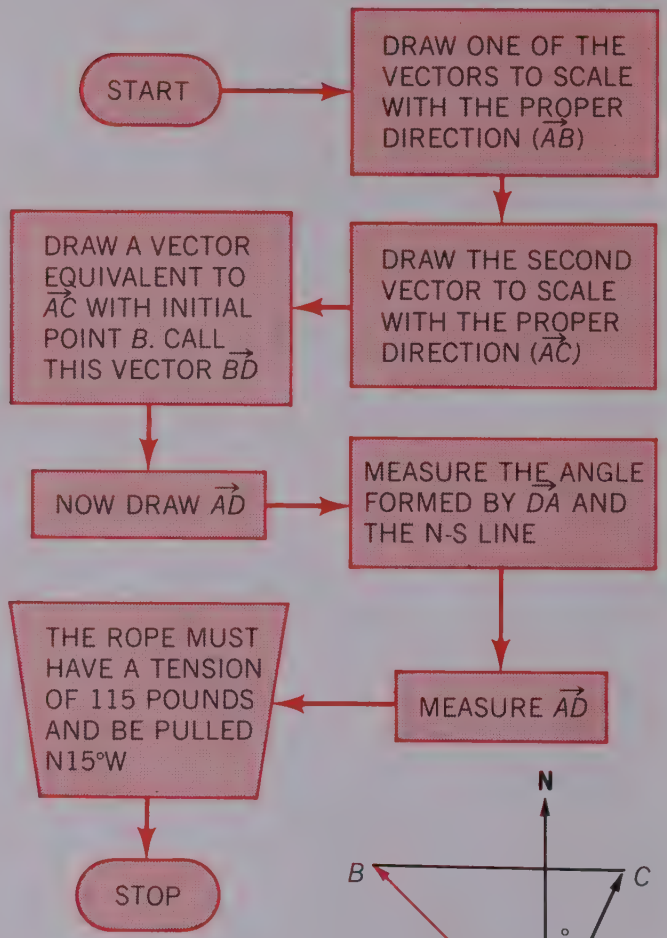
Figure 11-11

The tensions in the ropes $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are 80 pounds and 60 pounds respectively. If one rope were to replace the two ropes and have the same effect, in what direction and with what tension should the rope be pulled? (In vector terminology we ask "What is the *resultant* of the two vectors?")

Now we know that we cannot just draw to scale the diagram pictured and join the ends of the vectors (as shown in Figure 11-12) for two reasons:

1. The vector $\overrightarrow{BC}$ joins two terminal points (not according to the rule for addition).
2. Obviously the rope must be pulled northward not westward nor eastward.

Instead, we will replace one vector with an equivalent vector in the following way.


START
DRAW ONE OF THE VECTORS TO SCALE WITH THE PROPER DIRECTION ($\overrightarrow{AB}$)
DRAW THE SECOND VECTOR TO SCALE WITH THE PROPER DIRECTION ($\overrightarrow{AC}$)
DRAW A VECTOR EQUIVALENT TO $\overrightarrow{AC}$ WITH INITIAL POINT B. CALL THIS VECTOR $\overrightarrow{BD}$
NOW DRAW $\overrightarrow{AD}$
MEASURE THE ANGLE FORMED BY $\overrightarrow{DA}$ AND THE N-S LINE
MEASURE $\overrightarrow{AD}$
THE ROPE MUST HAVE A TENSION OF 115 POUNDS AND BE PULLED N15°W
STOP

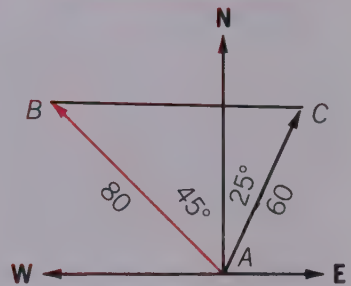


Figure 11-12

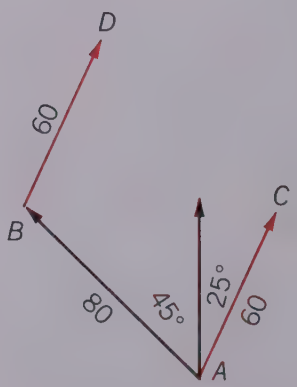


Figure 11-13

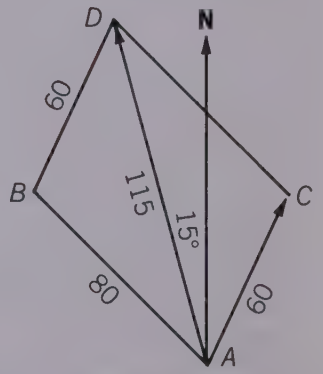


Figure 11-14

Although the problem has been completed there is something to be learned by studying the diagram further.

$$m(\overline{AC}) = m(\overline{BD})$$

and $$\overrightarrow{AC} \parallel \overrightarrow{BD}$$

Let us draw a vector with initial point C that is equivalent to $\overrightarrow{AB}$. The terminal point of this vector coincides with the terminal point of $\overrightarrow{BD}$, and we have

$$\overrightarrow{AB} \parallel \overrightarrow{CD}$$

and $$m(\overline{AB}) = m(\overline{CD})$$

Therefore figure $ABDC$ is a parallelogram and $\overrightarrow{AD}$ is the diagonal of the parallelogram. Our work suggests this rule:

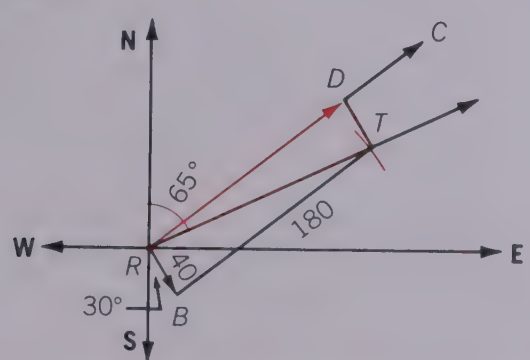
> If two vectors have common initial point (or a common terminal point), then, in the parallelogram that has these two vectors as adjacent sides, the diagonal containing the common point is the sum of the two vectors in both magnitude and direction.

Example 1

A pilot wishes to fly his airplane from Windsor to Toronto. Toronto is 200 miles N65°E of Windsor. The plane's air speed is 180 miles per hour. There is a wind from N30°W at 40 miles per hour. Find the course the pilot must set to arrive in Toronto. How long does it take him to fly to Toronto?

Solution

1. Draw the N-S, E-W lines. Label the point of intersection R.



2. Draw through R a ray to represent the direction of the track vector (N65°E).
3. Draw to scale the wind vector with R the initial point ($\overrightarrow{RB}$).
4. Use compasses with radius the scale measure of the magnitude of the air speed vector (180) and with centre B; describe an arc to cut the line of the track vector at T. Draw segment $\overrightarrow{BT}$.
5. Draw $\overrightarrow{RC}$ parallel to $\overrightarrow{BT}$ and $\overrightarrow{TD}$ parallel to $\overrightarrow{RB}$ with D on $\overrightarrow{RC}$.
6. $\overrightarrow{RD}$ is the course vector. Measure angle TRD (15°). This is the *wind correction angle.* Measure angle NRD (50°). $\angle NRD$ is the *navigation angle.*
7. $\overrightarrow{RT}$ represents the track vector. Measure to find its magnitude (176). This represents the plane's ground speed.
8. To find the time necessary to make the flight:

$$t = \frac{d}{r}$$

$$t = \frac{200}{176}$$

$$t = 1.14$$

Therefore, the pilot sets course N50°E.

The flight time is 1.14 hours approximately.

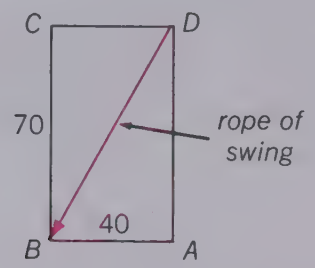
The device shown here is used to study forces through applications of vectors. Obtain this apparatus from your science lab. Repeat Project 1 and Project 2, pages 262-3, using this equipment.

Example 2

A child weighing 70 pounds is pushed in a swing with a horizontal force of 40 pounds. What is the tension in the rope? What is the angle of the rope to the vertical?

Solution

1. Draw a vector diagram.



   B is the seat of the swing. $\vec{AB}$ is the vector representing the 40 pound force. $\vec{CB}$ is the vector representing the 70 pound child.

2. Complete the parallelogram $ABCD$.

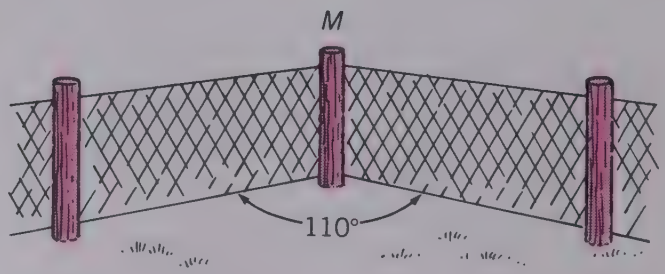
3. The sum of the vectors is the diagonal $\vec{DB}$. Note the direction of the vector.

4. By measurement (or by trigonometry): The magnitude of $\vec{DB}$ is approximately 80 and $m\angle BDA$ is approximately 30°. The rope of the swing is at an angle of 30° to the vertical. The tension in the rope is 80 pounds.

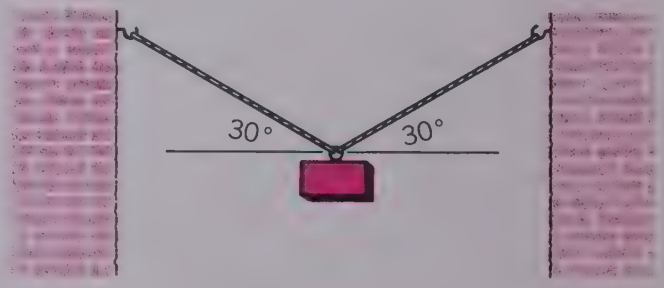
EXERCISE 11·6

1. A ball is suspended by two strings. The first is inclined 60° to the horizontal with a tension of 60 grams. The second is inclined 40° to the horizontal with a tension of 40 grams. What is the direction of and the tension in the one string that would hold the ball?

2. A corner post M supports a wire fence. One part of the fence runs S45°E while the other part runs S65°W. The tension in each is 110 pounds. Find the resultant tension on post M.



3. A weight is suspended from the middle of a cable between two buildings. Each half of this cable forms an angle of 30° with the horizontal. The maximum safe tension in the cable is 600 pounds. What is the maximum weight that can be supported safely?

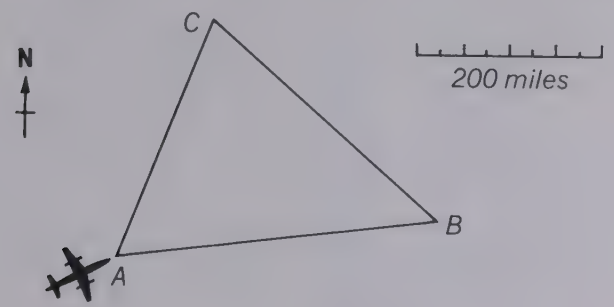


4. A pilot is calculating the course for his flight from Vancouver to Prince Rupert, a distance of 580 air miles. Prince Rupert is N40°W of Vancouver. The air speed of the plane is 160 miles per hour. There is a wind from S40°W at 50 miles per hour. Find the deviation of the course. How long does it take him to make the flight?

5. A fisherman sets out from camp to his favourite fishing lake. He hikes 3 miles N50°W, then 2.5 miles N30°E. How far is he from his camp and in what direction?

6. A plane with air speed of 150 miles per hour is set on a course of S43°E. There is a crosswind from S80°W at 45 miles per hour. Find the speed and the direction of the plane on its track.

7. Tell as much as you can from this scale drawing.



8. If three forces act on a body with no resulting motion, what can you say about the sum of the three vectors representing the forces?

Using the Library

1. *Life Science Library Mathematics* Read page 153. Explain why the man does not aim directly at the target.

   Read pages 182, 183, 184 and 185. Make a Mobius strip and cut it in halves as shown. What do you get? Make another Mobius strip and cut it in thirds as shown. Are you surprised at what you get?

   Can you make a map on a Mobius strip that requires six colors?

2. *The Wonderful World of Mathematics* Read the section "Stars and Steering" starting on page 26. Do you think a vector system can be used to help in navigating on the earth? Explain some of the problems encountered in navigating on the earth.

REVIEW EXERCISE 11

1. Name the two characteristics of a vector.

2. Two boys, Ted and John, were pulling on ropes attached to a sled. Ted pulled with a 60 pound force N30°W and John pulled with a 50 pound force N35°E.

   (a) Draw a vector diagram to scale.

   (b) Find the sum of the two force vectors.

   (c) Find the direction of the sum vector.

   (d) What one force could cause the same effect on the sled as the two forces? What is the direction of this one force?

3. John leaves home and walks 3 blocks east, 4 blocks south, 6 blocks east in order to reach his office. Find the distance and direction of John's office from his home.

4. A ship is set on course S10°E with water speed of 18 knots. A tide is running N40°E at 5 knots. Find the ship's actual (track) speed and the direction of its track.

5. A train travels due north at 60 miles per hour. Find the ground speed and the direction of the track of the conductor as he

   (a) walks towards the engine of the train at 6 miles per hour;

   (b) walks towards the rear of the train at 6 miles per hour;

   (c) walks directly west across the passenger coach at 6 miles per hour.

POWERS, ROOTS, AND RECIPROCALS 1–100

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
1	1	1	1.000	1.000	1.0000
2	4	8	1.414	1.260	.5000
3	9	27	1.732	1.442	.3333
4	16	64	2.000	1.587	.2500
5	25	125	2.236	1.710	.2000
6	36	216	2.449	1.817	.1667
7	49	343	2.646	1.913	.1429
8	64	512	2.828	2.000	.1250
9	81	729	3.000	2.080	.1111
10	100	1,000	3.162	2.154	.1000
11	121	1,331	3.317	2.224	.0909
12	144	1,728	3.464	2.289	.0833
13	169	2,197	3.606	2.351	.0769
14	196	2,744	3.742	2.410	.0714
15	225	3,375	3.873	2.466	.0667
16	256	4,096	4.000	2.520	.0625
17	289	4,913	4.123	2.571	.0588
18	324	5,832	4.243	2.621	.0556
19	361	6,859	4.359	2.668	.0526
20	400	8,000	4.472	2.714	.0500
21	441	9,261	4.583	2.759	.0476
22	484	10,648	4.690	2.802	.0455
23	529	12,167	4.796	2.844	.0435
24	576	13,824	4.899	2.884	.0417
25	625	15,625	5.000	2.924	.0400
26	676	17,576	5.099	2.962	.0385
27	729	19,683	5.196	3.000	.0370
28	784	21,952	5.292	3.037	.0357
29	841	24,389	5.385	3.072	.0345
30	900	27,000	5.477	3.107	.0333
31	961	29,791	5.568	3.141	.0323
32	1,024	32,768	5.657	3.175	.0312
33	1,089	35,937	5.745	3.208	.0303
34	1,156	39,304	5.831	3.240	.0294
35	1,225	42,875	5.916	3.271	.0286
36	1,296	46,656	6.000	3.302	.0278
37	1,369	50,653	6.083	3.332	.0270
38	1,444	54,872	6.164	3.362	.0263
39	1,521	59,319	6.245	3.391	.0256
40	1,600	64,000	6.325	3,420	.0250
41	1,681	68,921	6.403	3.448	.0244
42	1,764	74,088	6.481	3.476	.0238
43	1,849	79,507	6.557	3.503	.0233
44	1,936	85,184	6.633	3.530	.0227
45	2,025	91,125	6.708	3.557	.0222
46	2,116	97,336	6.782	3.583	.0217
47	2,209	103,823	6.856	3.609	.0213
48	2,304	110,592	6.928	3.634	.0208
49	2,401	117,649	7.000	3.659	.0204
50	2,500	125,000	7.071	3.684	.0200

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
51	2,601	132,651	7.141	3.708	.0196
52	2,704	140,608	7.211	3.733	.0192
53	2,809	148,877	7.280	3.756	.0189
54	2,916	157,464	7.348	3.780	.0185
55	3,025	166,375	7.416	3.803	.0182
56	3,136	175,616	7.483	3.826	.0179
57	3,249	185,193	7.550	3.849	.0175
58	3,364	195,112	7.616	3.871	.0172
59	3,481	205,379	7.681	3.893	.0169
60	3,600	216,000	7.746	3.915	.0167
61	3,721	226,981	7.810	3.936	.0164
62	3,844	238,328	7.874	3.958	.0161
63	3,969	250,047	7.937	3.979	.0159
64	4,096	262,144	8.000	4.000	.0156
65	4,225	274,625	8.062	4.021	.0154
66	4,356	287,496	8.124	4.041	.0152
67	4,489	300,763	8.185	4.062	.0149
68	4,624	314,432	8.246	4.082	.0147
69	4,761	328,509	8.307	4.102	.0145
70	4,900	343,000	8.367	4.121	.0143
71	5,041	357,911	8.426	4.141	.0141
72	5,184	373,248	8.485	4.160	.0139
73	5,329	389,017	8.544	4.179	.0137
74	5,476	405,224	8.602	4.198	.0135
75	5,625	421,875	8.660	4.217	.0133
76	5,776	438,976	8.718	4.236	.0132
77	5,929	456,533	8.775	4.254	.0130
78	6,084	474,552	8.832	4.273	.0128
79	6,241	493,039	8.888	4.291	.0127
80	6,400	512,000	8.944	4.309	.0125
81	6,561	531,441	9.000	4.327	.0123
82	6,724	551,368	9.055	4.344	.0122
83	6,889	571,787	9.110	4.362	.0120
84	7,056	592,704	9.165	4.380	.0119
85	7,225	614,125	9.220	4.397	.0118
86	7,396	636,056	9.274	4.414	.0116
87	7,569	658,503	9.327	4.431	.0115
88	7,744	681,472	9.381	4.448	.0114
89	7,921	704,969	9.434	4.465	.0112
90	8,100	729,000	9.487	4.481	.0111
91	8,281	753,571	9.539	4.498	.0110
92	8,464	778,688	9.592	4.514	.0109
93	8,649	804,357	9.644	4.531	.0108
94	8,836	830,584	9.695	4.547	.0106
95	9,025	857,375	9.747	4.563	.0105
96	9,216	884,736	9.798	4.579	.0104
97	9,409	912,673	9.849	4.595	.0103
98	9,604	941,192	9.899	4.610	.0102
99	9,801	970,299	9.950	4.626	.0101
100	10,000	1,000,000	10.000	4.642	.0100

TABLE OF SINES, COSINES, TANGENTS

Angle (Degrees)	Sine	Cosine	Tangent
0°	.0000	1.0000	.0000
1°	.0175	.9998	.0175
2°	.0349	.9994	.0349
3°	.0523	.9986	.0524
4°	.0698	.9976	.0699
5°	.0872	.9962	.0875
6°	.1045	.9945	.1051
7°	.1219	.9925	.1228
8°	.1392	.9903	.1405
9°	.1564	.9877	.1584
10°	.1736	.9848	.1763
11°	.1908	.9816	.1944
12°	.2079	.9781	.2126
13°	.2250	.9744	.2309
14°	.2419	.9703	.2493
15°	.2588	.9659	.2679
16°	.2756	.9613	.2867
17°	.2924	.9563	.3057
18°	.3090	.9511	.3249
19°	.3256	.9455	.3443
20°	.3420	.9397	.3640
21°	.3584	.9336	.3839
22°	.3746	.9272	.4040
23°	.3907	.9205	.4245
24°	.4067	.9135	.4452
25°	.4226	.9063	.4663
26°	.4384	.8988	.4877
27°	.4540	.8910	.5095
28°	.4695	.8829	.5317
29°	.4848	.8746	.5543
30°	.5000	.8660	.5774
31°	.5150	.8572	.6009
32°	.5299	.8480	.6249
33°	.5446	.8387	.6494
34°	.5592	.8290	.6745
35°	.5736	.8192	.7002
36°	.5878	.8090	.7265
37°	.6018	.7986	.7536
38°	.6157	.7880	.7813
39°	.6293	.7771	.8098
40°	.6428	.7660	.8391
41°	.6561	.7547	.8693
42°	.6691	.7431	.9004
43°	.6820	.7314	.9325
44°	.6947	.7193	.9657
45°	.7071	.7071	1.0000

Angle (Degrees)	Sine	Cosine	Tangent
45°	.7071	.7071	1.0000
46°	.7193	.6947	1.0355
47°	.7314	.6820	1.0724
48°	.7431	.6691	1.1106
49°	.7547	.6561	1.1504
50°	.7660	.6428	1.1918
51°	.7771	.6293	1.2349
52°	.7880	.6157	1.2799
53°	.7986	.6018	1.3270
54°	.8090	.5878	1.3764
55°	.8192	.5736	1.4281
56°	.8290	.5592	1.4826
57°	.8387	.5446	1.5399
58°	.8480	.5299	1.6003
59°	.8572	.5150	1.6643
60°	.8660	.5000	1.7321
61°	.8746	.4848	1.8040
62°	.8829	.4695	1.8807
63°	.8910	.4540	1.9626
64°	.8988	.4384	2.0503
65°	.9063	.4226	2.1445
66°	.9135	.4067	2.2460
67°	.9205	.3907	2.3559
68°	.9272	.3746	2.4751
69°	.9336	.3584	2.6051
70°	.9397	.3420	2.7475
71°	.9455	.3256	2.9042
72°	.9511	.3090	3.0777
73°	.9563	.2924	3.2709
74°	.9613	.2756	3.4874
75°	.9659	.2588	3.7321
76°	.9703	.2419	4.0108
77°	.9744	.2250	4.3315
78°	.9781	.2079	4.7046
79°	.9816	.1908	5.1446
80°	.9848	.1736	5.6713
81°	.9877	.1564	6.3138
82°	.9903	.1392	7.1154
83°	.9925	.1219	8.1443
84°	.9945	.1045	9.5144
85°	.9962	.0872	11.4301
86°	.9976	.0698	14.3007
87°	.9986	.0523	19.0811
88°	.9994	.0349	28.6363
89°	.9998	.0175	57.2900
90°	1.0000	.0000	undefined

Answers

CHAPTER 1

EXERCISE 1·1, Pages 2-3

2. (a) \$2,703,460 (b) \$2,703,500 (c) \$2,703,000 (d) \$2,700,000 (e) \$2,700,000
(f) \$3,000,000 **3.** \$4.71 **4.** 4.55, 0.38, 7.50, 41.46, 3.75
5. (a) 12.001", 1.864", 1.928" (c) 12.01", 1.87", 1.93"

EXERCISE 1·2, Pages 4-5

1. $x = \frac{FT}{T_s}$ **2.** (a) 20,000 (b) 19,360 **3.** (a) 336 **4.** 200 gal., 229 gal.
5. .0 697 mm **6.** (b) 7,510,000 (c) (i) 7,500,000 (ii) 9,000,000

EXERCISE 1·4, Page 10

1. (a) 4^5 (b) 5^7 (c) 15^9 (d) 10^4 (e) $2^6 \times 3^2$ (f) $15^0 = 1$ (g) a^4 (h) 2^{-6}
(i) 2^5 (j) $12^0 = 1$ (k) $14^0 = 1$ (l) 1 (m) 4^{-2} (n) 10^{-2} (o) $16^0 = 1$ (p) 4^6
(q) 8^8 (r) 1 (s) 5^{-4} (t) 4^{-2} **2.** (a) 2^9 (b) 2^8 (c) 2^6 (d) 2^2 (e) 2^5
(f) $2^0 = 1$ (g) 2^5 (h) 2^5 **3.** 128 **4.** 512 **5.** 1024 **6.** 1024 **7.** 32 **8.** 8 **9.** 16
10. 8 **11.** 625 **12.** 630 **13.** 660 **14.** 875 **15.** 1.7 **16.** 7 **17.** 12.5 **18.** 9.5 **19.** 4100
20. 16,000 **21.** 4100 **22.** 2.2 **23.** 29,000 **24.** 22 **25.** 10, 3744, 70, 491, 25,600

EXERCISE 1·5, Pages 13-15

1. (a) 3.42×10^2 (b) 6.724×10^3 (c) 7.23×10^1 (d) 1.4×10^{-1} (e) 5.627×10^2
(f) 2.00×10^2 (g) 2.000×10^2 (h) 4.7×10^0 (i) 1.4×10^{-3} (j) 3.9×10^{-2} (k) 9×10^{-4}
(l) 5.6×10^{-2} **2.** (a) 145 (b) 3240 (c) 5960 (d) 0.444 (e) 0.0576 (f) 0.00275
(g) 398.40 (h) 375.0 **5.** (a) 3.89 (b) 11.1 (c) 11.4 (d) 5.77 (e) 0.577 (f) 0.577
(g) 0.195 (h) 21.4 **6.** 4.55×10^6, 4,550,000 **7.** 4.30×10^6 **8.** 4.00×10^6
9. (a) 986 sq. ft. (b) \$1,233 **10.** 84.9 ft. **11.** \$61.90 **12.** (a) 1350 sq. mi. (b) 677
(c) 0.00148 (d) 19,900,000 acres (e) 1.06 (f) 0.945 (g) 1.2 acres

EXERCISE 1·6, Pages 17-18

7. (a) 570, 1330, 450, 1330, 1330, 1330, 1880, 1040, 430, 250, 760, 630, 430, 250, 230
(e) mean, 816; median, 630; mode, 1330

REVIEW EXERCISE 1, Page 20

1. (a) 1500 (b) 3500 (c) 8700 **2.** (a) 298,000 (b) 23.1 (c) 2.62 (d) 0.548
3. (b) mean, 4.4; median, 4.2; mode, 3.5.

CHAPTER 2

EXERCISE 2·1, Page 23

1. (a) T, $1 + 1 = 2 \in I$, $2 + 2 = 4 \in I$ (b) T, $2 \times 2 = 4 \in I$, $2 \times 3 = 6 \in I$
(c) T, $4 - 3 = 1 \in I$, $2 - 6 = -4 \in I$ (d) F, $2 \div 3 \notin I$
(e) T, $2 + 3 = 3 + 2$, $6 + 1 = 1 + 6$ (f) T, $2 \times 3 = 3 \times 2$, $6 \times 1 = 1 \times 6$
(g) T, $1 + (2 + 3) = (1 + 2) + 3$, $-1 + (3 + 6) = (-1 + 3) + 6$
(h) T, $1 \times (2 \times 3) = (1 \times 2) \times 3$, $2 \times (3 \times 6) = (2 \times 3) \times 6$
(i) F, $4 \div (2 \div 2) \neq (4 \div 2) \div 2$ **2.** 47,182 ft **3.** (a) 35 (b) 6 (c) –81
(d) 29 (e) 654 (f) –312 **4.** (a) –120 (b) 1050 (c) 9 (d) 27

EXERCISE 2·2, Page 24

1. Rational nos: (a), (b), (c), (e), (f), (i), (j), (k), (n), (o) **2.** 0, 2, 3, 5, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{1}{3}$, $\frac{5}{6}$
3. $-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}$, 0, $-\frac{3}{5}, -\frac{2}{5}, -\frac{1}{5}, -\frac{1}{3}, -\frac{1}{6}$ **4.** $\ldots, -2, -1, 0, 1, 2, \ldots$, $V = I$ **5.** $\{0\}$

EXERCISE 2·3, Page 25

1. $\frac{2}{3}$ **2.** $\frac{3}{4}$ **3.** $\frac{2}{5}$ **4.** $\frac{9}{14}$ **5.** $\frac{3}{5}$ **6.** $\frac{11}{63}$ **7.** $\frac{5}{7}$ **8.** $\frac{1}{3}$ **9.** $\frac{2}{9}$ **10.** $\frac{13}{22}$ **11.** $\frac{32}{33}$
12. $\frac{6}{7}$ **13.** $\frac{3}{11}$ **14.** $\frac{5}{7}$ **15.** $\frac{2}{3}$ **16.** (a) $\frac{6}{10}, \frac{12}{20}, \frac{9}{15}, \frac{18}{30}$ (b) $\frac{10}{12}, \frac{20}{24}, \frac{15}{18}, \frac{30}{36}$
(c) $\frac{6}{16}, \frac{12}{32}, \frac{9}{24}, \frac{18}{48}$ (d) $\frac{14}{24}, \frac{28}{48}, \frac{21}{36}, \frac{42}{72}$ **17.** (a) $\{\frac{2}{7}, \frac{6}{21}, \frac{8}{28}, \frac{4}{14}, \ldots\}$
(b) $\{\frac{3}{13}, \frac{6}{26}, \frac{12}{52}, \frac{15}{65}, \frac{9}{39}, \ldots\}$ (c) $\{\frac{4}{9}, \frac{40}{90}, \frac{36}{81}, \frac{320}{720}, \ldots\}$ (d) $\{\frac{4}{15}, \frac{16}{60}, \frac{24}{90}, \frac{32}{120}, \frac{20}{75}, \ldots\}$
18. $\frac{1}{9}$ **19.** 0.41, 3.2×10^9

EXERCISE 2·4, Pages 26-27

1. (a) $\frac{14}{27}$ (b) $\frac{18}{-35}$ (c) $\frac{-26}{45}$ (d) $\frac{-42}{85}$ (e) $\frac{-6x}{35}$ (f) $\frac{10xy}{21a}$ (g) $\frac{-21}{y}$ (h) b (i) $\frac{xy}{zm}$ (j) $\frac{5x}{18}$

2. (a) −2 (b) −4 (c) −7 (d) −3 (e) 7 (f) 11 (g) −9 (h) −3 **3.** (a) $\frac{10}{15}$ (b) $\frac{9}{21}$

(c) $\frac{32}{72}$ (d) $\frac{-9}{24}$ (e) $\frac{-4}{1}$ (f) $\frac{-2}{1}$ (g) $\frac{9}{b}$ (h) $\frac{4}{1}$ (i) $\frac{-2}{1}$ (j) $\frac{-1}{3}$ (k) $\frac{-ac}{bc}$ (l) $\frac{xy}{zy}$

(m) $\frac{a}{b}$ (n) $\frac{k}{-1}$ (o) $\frac{17}{-1}$ (p) $\frac{-1}{r}$ **4.** (a) $\frac{-8}{63}$ (b) $\frac{-7}{26}$ (c) $\frac{1}{6}$ (d) $\frac{1}{2}$ (e) $-\frac{6}{7}$ (f) $-\frac{6}{7}$

EXERCISE 2·5, Page 27

1. (a) $\frac{7}{4}$ (b) $\frac{-2}{-3}$ (c) $\frac{2}{3}$ (d) $\frac{-3}{2}$ (e) $\frac{1}{2}$ (f) $\frac{1}{n}$ (g) $\frac{1}{c}$ (h) $\frac{b}{a}$ **2.** (a) 4 (b) 7 (c) 5 (d) 5 (e) −8 (f) −7

3. (a) $-\frac{6}{7}$ (b) $-\frac{5}{6}$ (c) $-\frac{21a}{32}, b \neq 0$ (d) $\frac{13b}{2772}$ (e) $-\frac{2x}{7y^2}, y \neq 0$ (f) $\frac{7n}{4x}, x \neq 0$

EXERCISE 2·6, Page 29

1. (a) 16 (b) 9 (c) 3 (d) −16 (e) 24a (f) 14 (g) −3a (h) 4a (i) 6b
(j) −24c (k) 0 (l) 36 **2.** (a) $\frac{1}{5}$ (b) $\frac{4}{9}$ (c) $\frac{13}{a}$ (d) $\frac{p+2a}{b}$ (e) $\frac{9}{8}$ (f) $\frac{3}{4}$

(g) $\frac{-8}{21}$ (h) $\frac{11}{9}$ (i) $\frac{9}{5}$ (j) $\frac{2a+c}{8}$ (k) −7 (l) $\frac{51}{77}$ (m) $\frac{21}{4a}$ (n) $\frac{6a-3}{ax}$

(o) $\frac{10}{21}$ (p) $\frac{23}{21}$ (q) $\frac{4a+3}{ax}$ (r) $\frac{a+b}{ab}$ (s) $\frac{3y+14x}{xy}$ (t) $\frac{2d+3b}{bd}$

3. (a) $\frac{n}{mn} + \frac{-m}{mn} = \frac{n-m}{mn}$ (b) $\frac{2x}{ax} + \frac{-3a}{ax} = \frac{2x-3a}{ax}$ (c) $\frac{4d}{bd} + \frac{3b}{bd} = \frac{4d+3b}{bd}$

(d) $\frac{3mr}{pr} + \frac{-2np}{pr} = \frac{3mr-2np}{pr}$ (e) $\frac{ys}{xs} + \frac{xz}{xs} = \frac{ys+xz}{xs}$ (f) $\frac{35}{30} + \frac{-24}{30} = \frac{11}{30}$

4. $16\frac{1}{3}$ in, $25\frac{1}{2}$ in, $22\frac{5}{6}$ in, $110\frac{1}{4}$ cu. in.

EXERCISE 2·7, Page 30

1. (a) $\frac{4}{5} + (-\frac{2}{3})$ (b) $\frac{3}{5} + (-\frac{1}{3})$ (c) $\frac{4a}{3} + \frac{a}{3}$ (d) $\frac{3}{14} + \frac{1}{2a}$ **2.** (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) $\frac{1}{3}$

(e) $\frac{35}{8a}$ (f) $\frac{12b+14c}{21c}$ (g) $\frac{14x+3y}{4y}$ (h) $\frac{12x+3y}{4xy}$ (i) $\frac{ad-bc}{bd}$ (j) $\frac{37}{15a}$

3. $\frac{1}{6}, \frac{29}{18}, \frac{347}{60}, \frac{15}{16}, \frac{2}{15}, \frac{25}{32}, \frac{25}{18}, \frac{9}{5}, \frac{25}{8}, \frac{817}{400}$ **4.** $b = 2$; $c = 2$, $d = 1$; $d = 0$; $m = 3$, $n = 2$;
$x = \frac{1}{2}$; $x = 1$, $y = -1$; $a = b = \pi$ **5.** $\frac{4}{3}, \frac{52}{111}, \frac{9}{5}, \frac{8}{3}$ **6.** (a) 1730 (b) $12,654,250,000 (c) $594
(d) $2,610

EXERCISE 2·8, Page 33

1. (a) closure under mult. (b) closure under div. (c) dist. (d) mult. prop. of 1 (e) mult. prop. of inverses (f) comm. under add. (g) add. of inverses (h) comm. under mult. (i) add. prop. of 0 (j) closure under add. (k) closure under subt. **3.** (a) APE (b) SPE (c) MPE (d) mult. of inverses (e) TPE (f) add. of inverses (g) SPE (h) comm. mult. (i) dist. (j) comm. add. (k) MPE (l) mult. of inverses (m) APE (n) add. of inverses (o) TPE (p) APE (q) assoc. add. (r) mult. of inverses (s) APE (t) add. of inverses (u) MPE (v) mult. of inverses

5. 300 tons; 3,600,000; $564,000

EXERCISE 2·9, Page 35

1. (a) 4 (b) $4\frac{2}{5}$ (c) $-4\frac{2}{3}$ (d) 1 (e) 10 (f) −32 (g) $\frac{1}{3}$ (h) 6 (i) −4 (j) $-\frac{1}{3}$ (k) $\frac{3}{4}$ (l) $-6\frac{2}{5}$ (m) $-2\frac{3}{5}$ (n) $-8\frac{1}{8}$ (o) $1\frac{3}{11}$ (p) $2\frac{1}{4}$ **2.** 14.4; 100.5; 5240

EXERCISE 2·10, Pages 37-38

1. 0.8 **2.** 0.25 **3.** 0.125 **4.** 0.0625 **5.** 0.3125 **6.** 0.09375 **7.** 1.171875 **8.** 0.0234375
9. $0.\dot{3}$ **10.** $0.1\dot{6}$ **11.** $0.\dot{1}$ **12.** $0.\dot{6}$ **13.** $0.8\dot{3}$ **14.** $0.\dot{4}$ **15.** 0.59375
16. 1.59375 **17.** $0.4\dot{5}$ **18.** $0.\dot{0}76923\dot{0}$
19. $0.\dot{1}53846\dot{1}$ **20.** $0.\dot{0}\dot{1}$ **21.** $0.\dot{1}42857, 0.2\dot{8}5714, 0.4\dot{2}8571\dot{4}$ **22.** 047619 **23.** 428571 **24.** 857142
25. 238095 **26.** $3.5\dot{7}1428\dot{5}$ **27.** $3.33, 405\frac{2}{3}$

EXERCISE 2·11, Pages 38-39

1. $10, 16, 12, 28$ **2.** $6, 12, 18, 22$ **3.** $6, 12, 16, 18, 46$ **4.** $0.0\dot{9}, 0.1\dot{8}, 0.2\dot{7}, 0.3\dot{6}, 0.4\dot{5}, 0.5\dot{4}, 0.6\dot{3}, 0.7\dot{2}, 0.8\dot{1}, 0.\dot{9}0$; 2; 5 **5.** $0.\dot{0}76923, 0.1\dot{5}3846, 0.2\dot{3}0769, 0.3\dot{0}7692, 0.3\dot{8}4615\dot{3}$ $0.4\dot{6}1538, 0.5\dot{3}8461, 0.6\dot{1}5384, 0.6\dot{9}2307, 0.7\dot{6}9230, 0.8\dot{4}6153, 0.9\dot{2}3076$; 6; 2

EXERCISE 2·12, Page 40

1. (a) $\frac{2}{3}, \frac{5}{7}$ (b) $\frac{4}{9}, \frac{5}{11}$ (c) $\frac{33}{65}, \frac{11}{15}$ (d) $\frac{14}{71}, \frac{41}{53}$ (e) $\frac{3}{7}, \frac{13}{29}, \frac{10}{22}$ (f) $\frac{21}{44}, \frac{32}{65}$ **2.** 900,000 cu. ft., 6.92 yrs.

EXERCISE 2·14, Pages 41-42

Repeating decimals: **2,4** **7.** 12.124 **8.** 5.692 **9.** 22.136 **10.** 31.623 **11.** 1.863
12. 6.116 **13.** 19.339 **14.** 61.156 **15.** 0.612 **16.** 0.193 **17.** 0.061 **18.** 0.019
19. (a) 1 (b) 8 (c) 27 (d) 64 (e) 125 (f) 1000 (g) −27 (h) −8 **21.** (a) 1, 2, −1, −2, 4, −3 (b) ±1, ±4, ±2, ±5, ±10, ± 20 (c) −1, 1, 2, −2, 5, 20 **23.** (a) 3,4 (b) 4,5 (c) −4,−5 (d) 5,6 (e) 9,10 (f) 9,10 (g) 7,8 (h) 17, 18 (i) 56, 57 **24.** (a) 21.0 (b) 47.9 **25.** 9.80 **26.** (a) 4.32×10^6 cal. (b) 0.416 min. **27.** 5.27 amp. **28.** First generates $2\frac{1}{4}$ times as much heat.

EXERCISE 2·15, Page 44

1. (a) $0.12112111211112111112111111112$ (b) $0.12122122212222122222212222221$
(d) $0.476547655476555476555547655555$ (e) $0.12123123412345123456123456 7$

EXERCISE 2·16, Page 45

2. (a) 2144.7 (b) 2144.9 (c) 2143.6 **3.** 143.5 weeks

EXERCISE 2·18, Page 50

1. 3.00 **2.** 2.64 **3.** 21.2 **4.** 12.2 **5.** 25.2 **6.** 22.9 **7.** 8.00 **8.** 9.21 **9.** 8.72 **10.** 5.74
11. 35.3 **12.** 40 **13.** 41.6 **14.** 67.5 **15.** 85.2 **16.** 2.00 **17.** 1.71 **18.** 3.00 **19.** 4.33 **20.** 4.02
21. 3.56 **22.** 5.00 **23.** 7.49 **24.** 8.65 **25.** 9.54 **26.** 12.0 **27.** 20.0 **28.** 30.0
29. (a) 4.90 (b) 4.24 (c) 7.06 (d) 24.0 (e) 38.1 (f) 77.7 (g) 55.6 (h) 174
30. (a) 121 (b) 178 (c) 204 (d) 796 (e) 2140 (f) 1810 (g) 4230

EXERCISE 2·19, Pages 50-52

1. (a) 4.69 (b) −2.72 (c) −3.0 (d) 36.6 (e) 3.77 (f) 4.38 **2.** (a) 10.2 ft (b) 40.8 ft (c) 4.80 min **3.** 19.3 ft **4.** 6.48, 2.83, 6.25, 9.49, 11.0, 12.2, 0.600, 1.90, 6.00, 19.0, 60.0 **5.** 3.48, 2.00, 3.39, 4.48, 4.95, 5.31, 0.711, 1.53, 3.30, 7.11, 15.3 **6.** 1.732, 5.292, 6.856, 7.416, 8.000, 9.899 **7.** 1.442, 3.037, 3.609, 3.803, 4.000, 4.610 **8.** (a) $\sqrt{36}=6$, $\sqrt{47}=6.856$, $\sqrt{56}=7.483$ (b) 3.302, 3.609, −1.36, −1.587, −1.306, 3.826 **9.** $-\sqrt{49}=-7$, $-\sqrt{144}=-12$, $\sqrt{3^2}=3$, $\sqrt{296}=17.2$, $\sqrt[3]{16}=4$, $\sqrt[4]{296}=4.15$, $\sqrt[3]{125}=5$, $\sqrt[3]{-125}=-5$, $\sqrt[3]{729}=9$, $\sqrt[4]{16}=2$ **10.** (a) 65.57 (b) 6.20 (c) 29.60 (d) 0.24 (e) 0.75 (f) 2.37 (g) 7.48 (h) 23.66 (i) 74.83 **11.** 510 sq. ft. **12.** $\frac{1}{9}$ **13.** $\frac{1}{27}$ **14.** 27 **15.** (a) 3.14159 (b) 3.141592

EXERCISE 2·20, Pages 53-54

1. 9 **2.** 20 **3.** 42 **4.** 40 **5.** 110 **6.** 63 **7.** 180 **8.** 220 **9.** 24 **10.** 1080
11. 1100 **12.** 4800 **14.** $\sqrt{21}$ **15.** $\sqrt{66}$ **16.** $\sqrt{330}$ **17.** $\sqrt{350}$ **18.** $\sqrt{91}$
19. $\sqrt{465}$ **20.** $\sqrt{60}$ **21.** $\sqrt{510}$ **22.** $\sqrt{231}$ **23.** $\sqrt{130}$ **24.** $\sqrt{1430}$ **25.** $\sqrt{210}$
27. (a) 60 (b) $\frac{1}{2}\sqrt{126}$ (c) $3\sqrt{12}$ (d) $2\sqrt{88}$ (e) $\frac{1}{2}\sqrt{408}$ (f) 7

EXERCISE 2·21, Pages 54-55

1. $3\sqrt{2}$ **2.** $5\sqrt{2}$ **3.** $5\sqrt{3}$ **4.** $3\sqrt{7}$ **5.** $6\sqrt{3}$ **6.** $10\sqrt{2}$ **7.** $8\sqrt{2}$ **8.** $12\sqrt{2}$ **9.** $7\sqrt{2}$
10. $11\sqrt{2}$ **11.** $5\sqrt{7}$ **12.** $5\sqrt{14}$ **13.** $2\sqrt{2}$ **14.** $2\sqrt{7}$ **15.** $9\sqrt{5}$ **16.** $10\sqrt{3}$ **17.** $\sqrt{22}$
18. $a\sqrt{ab}$ **19.** $ab\sqrt{a}$ **20.** $a^2b\sqrt{b}$ **21.** $abc\sqrt{c}$ **22.** $abc\sqrt{abc}$ **23.** $2a\sqrt{a}$ **24.** $2b\sqrt{5bc}$

EXERCISE 2·22, Pages 56-57

1. $37\sqrt{3}$ **2.** $6\sqrt{11}$ **3.** $4\sqrt{5}$ **4.** $11\sqrt{7}-2\sqrt{13}+11\sqrt{3}$
5. $14\sqrt{13}-14\sqrt{17}$ **7.** 0 **8.** $\sqrt{41}-12\sqrt{51}$ **9.** $-2\sqrt{3}$
10. $14\sqrt{2}-18\sqrt{3}$ **11.** $4\sqrt{5}$ **12.** $-4\sqrt{3}$ **13.** $6\sqrt{13}$ **14.** $24-8\sqrt{2}$ **15.** $-\sqrt{5}$ **16.** $2\sqrt{2}+10\sqrt{3}$

EXERCISE 2·23, Pages 57-58

11. 36 **12.** 300 **13.** 70 **14.** $6a^2\sqrt{2b}$ **15.** $16ab^2\sqrt{2}$ **16.** $81a^2\sqrt{3}$ **17.** $\sqrt{15}-\sqrt{5}$
18. $7+3\sqrt{7}$ **19.** $2\sqrt{3}-3\sqrt{2}$ **20.** $\sqrt{xy}-x+\sqrt{xw}$ **21.** $x+y+2\sqrt{xy}$ **22.** $38-12\sqrt{10}$
23. $5\sqrt{6}-10\sqrt{15}$ **24.** $18-7\sqrt{6}$ **25.** $5\sqrt{2}+3\sqrt{10}-2\sqrt{5}-6$ **26.** −1 **27.** −8 **28.** −19 **29.** 15
30. $(89+24\sqrt{5})$ sq. in. **31.** $(3\sqrt{6}+3\sqrt{10})$ sq. in. **32.** $\sqrt{4}+\sqrt{6}$ **33.** 100 sq. rods

EXERCISE 2·24, Pages 58.59

1. 2.450 **2.** 2.828 **3.** 3.162 **4.** 3.873 **5.** 1.342 **6.** 8.660 **7.** 7.746 **8.** 5.477
9. 4.899 **10.** 5.196 **11.** 3.146 **12.** 5.278 **13.** 24.177 **14.** 3.464 **15.** −2.368 **16.** 5.782
17. 10.392 **18.** 12.250 **19.** 58.095 **20.** (a) 11.2 rods, 8.94 rods (b) 100 sq. rods (c) 40.3 rods **21.** $10\sqrt[3]{\frac{9}{4}}$ ft **22.** 3 gal. **23.** 80 ft

REVIEW EXERCISE 2, Page 60

1. $\{\sqrt{2}, -\sqrt{2}\}$ **2.** (a) $\frac{2}{3}$ (b) $\frac{2}{3}$ **3.** (a) $8\sqrt{6}$ (b) 0 (c) $\sqrt{3}$
4. 22.4 **5.** 5.83 **8.** 2540 sq. in. **9.** 1160 cu. ft.

CHAPTER 3

EXERCISE 3·1, Pages 63-64

1. (a) {2, 3, 7, 11}, {3, 4, 8, 13}, (b) {1, 3, 5, 7, ...}, {3, 5, 7, 9, ...} (c) {2, 4, 6, 8, ...}, {6, 8, 10, 12, ...} (d) {0, 1, 2, 4, ...}, {0, 1, 4, 16, , ...} **2.** (a) $\{x \in I^+ | x < 9\}$, $\{y | y = x^3, x \in I^+, x < 9\}$ (b) I, $\{y | y = x^3, x \in I\}$ (c) R, $\{w | w = 3v + 2, v \in R\}$

5. (Cadillac, Mercury), (Cadillac, Mercedes), (Cadillac, Vega), (Cadillac, Volkswagen), (Mercury, Mercedes), (Mercury, Vega), (Mercury, Volkswagen), (Mercedes, Vega), (Mercedes, Volkswagen), (Vega, Volkswagen)
6. (*A*,*B*), (*C*,*A*), (*C*,*D*), (*C*,*E*), (*E*,*F*), (*E*,*G*); *C* won; runners-up, *A*, *D*, and *E* **7.** 6 **9.** (a) (2,−3), (2,−1), (2,0), (5,2), (5,−3), (5,−1), (5,0), (5,3), (−1,−3), (0,−1), (0,−3), (3,2), (3,−3), (3,−1), (3,0) (b) (−3,2), (−1,−2), (0,2), (2,5), (−3,5), (−1,5), (0,5), (3,5), (−3,−1), (−1,0), (−3,0), (2,3), (−3,3), (−1,3), (0,3) (c) (0,0) (d) (−3,3), (3,−3) (e) (0,−1), (3,2)

EXERCISE 3·2, Pages 66-67

1. 12 kinds **2.** (a) (1, $9,000), (2, $20,000), (3, $25,000), (4, $30,000) (b) ($9,000, 1), ($20,000, 2), ($25,000, 3), ($30,000, 4) **4.** 15 **6.** {(*x*,1), (*x*,2), (*y*,1), (*y*,2), (*z*,1), (*z*,2)} (a) 6 (b) 2 **7.** {(5,5), (5,6), (5,7), (6,5), (6,6), (6,7), (7,5), (7,6), (7,7)}, 9 **8.** (a) 16 (b) 12 (c) 12 (d) 9 **9.** 35, 25, 20, 28
10. {(3,1), (5,3), (12,10)} **12.** {(5,4), (5,2), (5,6), (3,4), (3,2), (3,6), (7,4), (7,2), (7,6), (1,4), (1,2), (1,6)}
(a) {(5,6), (3,4), (3,6), (1,4), (1,2), (1,6)} (b) {(5,4), (5,2), (3,2), (7,4), (7,2), (7,6)} (c) {(5,2)} (d) ϕ
(e) {(1,4), (5,6)}

EXERCISE 3·3, Pages 69-70

2. (b) {(0,−2), (1,1), (2,4), (3,7), (4,10)} **4.** (a) YOU DID WELL. TRY THE NEXT ONE.
(b) Make a message and exchange it with a friend. **7.** {−13, −19, −25, −29, −49}
8. {−8, −9, −12, −15, −22, −47}
10. $h \to 6h$ (a) 24 sq. in. (b) 60 sq. in. (c) 270 sq. in. (d) $52\frac{1}{2}$ sq. in. **11.** $r \to \pi r^2$ (a) 49π (b) 196π (c) 900π

EXERCISE 3·4, Pages 72-73

1. 4, 2, −1, −7; −5, −2, 1, 4 **4.** (a) (5,3) (b) (0,3) (c) (−2,1) (d) (−3,−2) (e) (−1,0) (f) (1,−2) (g) (6,0) **5.** (a) $(4,2\frac{1}{2})$ (b) $(0,2\frac{1}{2})$ (c) $(-3,1\frac{1}{2})$ (d) (−4,0) (e) (−3,−3) (f) $(0,-3\frac{1}{2})$ (g) $(3,-2\frac{1}{2})$

EXERCISE 3·5, Pages 77-79

1. function, (1,6), (2,9), (3,7), (4,8), (5,10); function, (*m*,3), (*n*,3), (*o*,3), (*p*,4), (*q*,4); not a function **2.** (a) {2,4,12,18} (b) {4,6,14,16} (c) one **3.** (a) {100,95,90,85,75,70} (b) {1,2,3,4} (c) one (d) yes **4.** (a) {9,16,25,36,4,1} (b) {3,4,−5,−6,−3,−2,−1} (c) is not, because 9 is mapped into 3 and −3. **5.** (a) no (b) yes (c) yes (d) no (e) no
6. functions, (a), (b), (e), (i), (j), (l) **7.** (a) no (b) no (c) no (d) no (e) yes (f) no (g) yes (h) yes (i) yes (j) yes (k) no **8.** (a) 10¢, 20¢, 45¢, 65¢ **9.** 28 mi.

EXERCISE 3·6, Page 80

1. (a) $x \to 3x$ (b) $x \to x - 3$ (c) $x \to 5x + 13$ (d) $x \to 5(x + 13)$
(e) $x \to \sqrt{x}$ (f) $x \to x^2 + \pi$ (g) $x \to 3(x^2 + 5)$ **3.** (a) 14.8 (b) 16.3

EXERCISE 3·7, Pages 82-83

2. (a) 4,3 (b) 3,−2 (c) no *x*-intercept, 2 (d) 0,0 (e) −2,−4 (f) −3,2
3. *A*(4,0), *B*(0,3), *C*(0,−2), *D*(3,0), *E*(0,2), *F*(0,0), *G*(−2,0), *H*(0,−4)

EXERCISE 3·8, Page 84

1. (a) yes (b) yes (c) yes (d) yes, yes (e) no (f) yes **2.** 3,−4 **3.** $\frac{41}{2}, \pi$ **4.** −3,0
5. $\sqrt{3}, -\sqrt{3}$ **6.** $0, -\sqrt[3]{6}, -\frac{4}{3}$ **7.** $\sqrt{5}$,0 **8.** $y = -x + 4$ **9.** $y = x - 14$ **10.** $y = 3x + 5$
11. $y = 2x - 3$ **12.** $y = 2x + \frac{5}{2}$ **13.** $y = -\frac{3}{2}x - \frac{7}{2}$ **14.** $y = x + 1$ **15.** $y = \frac{1}{2}x - \frac{1}{3}$ **16.** $y = -2x + 7$
17. $y = x + \frac{3}{2}$ **18.** $y = \frac{5}{4}x - 7$ **19.** $y = x + \frac{6}{5}$

EXERCISE 3·9, Pages 86-87

1. increases, 5, increases, 15, decreases, 10,0, multiplier
2. 0, decreases, 3, decreases, 12, increased by 9 units, multiplier **3.** 4, remains 4, remains 4, constant

EXERCISE 3·10, Page 89

1. increases, 4, decreases, 8, decreases, 16, −5, (0,−5), 8, (2,3) **3.** $330,525; 147 days

EXERCISE 3·12, Pages 92-95

2. (a) rise, positive (b) fall, negative (c) rises, falls (d) c and f, 2 and -2 (e) a and d, $\frac{1}{4}$ and $-\frac{1}{4}$ (f) gentle, steep (h) 3,6 (i) 6 (j) 3 (k) $(\overline{MP})$ (n) $-4, 8$ (o) -2 (p) equal (q) 1 (r) $\frac{1}{4}, -\frac{1}{4}, -1$ **3.** $5, -\frac{1}{2}$ **4.** $A, \frac{1}{4}$; $B, \frac{5}{8}$; $C, \frac{5}{4}$; $D, \frac{18}{8}$ **5.** A, 1; B, 3; C, -7; D, -1; E, $-\frac{1}{4}$ **7.** $0, y = 0x + 3$ **8.** (a) $2, -3, -\frac{5}{2}$ (b) $-2, 8, 0$ **9.** (a) $\{(x,y)|y = 3x + 2\}$ (b) $\{(x,y)|y = \frac{1}{4}x + 10\}$ (c) $\{(x,y)|y = 0\}$ (d) $\{(x,y)|y = x + 1\}$ **10.** $3, -\frac{9}{10}, 2, -1$ **11.** (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) $-\frac{2}{3}$ (e) $-\frac{1}{4}$ (f) $\frac{4}{7}$ (g) $\frac{5}{2}$ **14.** $\frac{4}{9}$ **15.** (a) $\frac{4.5}{100}, \frac{2.2}{100}$ (b) 1.7, 1.6

EXERCISE 3·14, Pages 97-99

1. (4,5) **2.** (a) (7,4) (b) (5,−5) (c) (−5,0) (d) (−2,6) (e) $(-\frac{8}{5}, 4)$ (f) $(-\frac{19}{5}, -\frac{6}{5})$ **3.** (a) empty (b) (iv) parallel (vii) ϕ **4.** $(\frac{9}{4}, \frac{11}{4})$ (c) (i) $(\frac{3}{2}, \frac{5}{2})$ (ii) $(\frac{16}{3}, \frac{17}{3})$ **5.** 5 ft, 4 ft **6.** $Y = 12$ gal, $X = 8$ gal **7.** 9,13 **8.** Jill, 3 ft; Judy, 6 ft

EXERCISE 3·15, Page 100

1. (a) $(4\frac{1}{3}, 22)$ (b) $(2\frac{2}{5}, -3\frac{3}{5})$ (c) $(3\frac{2}{11}, 26\frac{2}{11})$ (d) $(9, -16)$ (e) $(2\frac{6}{7}, 18\frac{5}{7})$ (g) (−10, 14) (h) (1.8, 140)

EXERCISE 3·17, Pages 102-104

1. $\$0.075x$ **2.** $\$0.52x$, $\$0.43y$, $\$(0.52x + 0.43y)$ **3.** $(2x + 2y)$ inches **4.** 260 mph, 1040, 390, 2340, 3380 mi, 338 mph **5.** $4x + 4y$ **6.** $35x + 12y$ **7.** $f = 3t$, $5f + 10t = 175$ **8.** $x - 3y = 20$, $x + y = 90$ **9.** $w + f =$ total yards required, $2.80w + 1.40f = 840$ **10.** $x + 3y = 7.05$, $x + 7y = 9.65$ **11.** $x + y = 180$, $\frac{2}{5}x = \frac{1}{2}y$ **12.** $x + y = 25$, $\pi x - \pi y = 9.42$ **13.** $x + y = 40$, $x = \frac{3}{2}y$ **14.** $a - b = 100$, $\frac{3}{2}(a - 30) + \frac{3}{2}(b + 30) = 900$ **15.** \$600 at each rate **16.** \$825 at 9%, \$1175 at 7% **17.** 40 lb, 60 lb **18.** 82 in, 64 in **19.** \$1933.33, \$866.67 **20.** 110 yd cement, 550 yd sand and stone **21.** 1540 gal at 40¢, 460 gal at 50¢

REVIEW EXERCISE 3, Page 104

1. (a) {(31, Jan.), (28, Feb.), (31, Mar.), (30, Apr.), (31, May), (30, June), (31, July), (31, Aug.), (30, Sept.), (31, Oct.), (30, Nov.), (31, Dec.)} (b) no **2.** $y = -\frac{1}{3}x + \frac{2}{3}$ **4.** (−4, 6) **5.** $(2\frac{1}{2}, 4\frac{1}{2})$ **6.** $h + l = 170$, $700h + 1200l = 175{,}000$ **7.** $\frac{3}{2}, -4$

CHAPTER 4

EXERCISE 4·2, Pages 108-109

1. \$49.84 **2.** \$50.84 **3.** \$2.10 **4.** (a) \$57.60 (b) \$1.85 (c) \$7.68 (d) \$15.40 (e) \$15.10

5. 9% **6.** $6\frac{1}{2}\%$ **7.** \$250 **8.** \$18.75 **9.** (a) $r = \frac{I}{Pt}$ (b) $t = \frac{I}{Pr}$ (c) $P = \frac{I}{rt}$

EXERCISE 4·3, Page 111

8. \$28.75 **9.** \$336

EXERCISE 4·4, Page 113

3. \$79.35 **4.** (a) \$121.14 (b) \$67.72 (c) \$139.35 (d) \$91.38

EXERCISE 4·5, Pages 113-114

1. \$63 **2.** (a) \$48.38 (b) \$57.50 (c)\$107.50(d) \$60.50

EXERCISE 4·6, Page 115

1. \$2.10, apparent rate = 5%, true rate = 8.4% **2.** 46.4%, 79.6%; 9.6%, 16.9%; 18.4%, 33.9%

EXERCISE 4·7, Page 117

1. 64.5% **2.** (a) \$100 loan: 11.9%, 12.1%, 12.3%, 12.5% \$600 loan: 12.0%, 12.3%, 12.5%, 12.6% (b) 12 months: 12.0%, 12.0%, 12.0%, 12.0% 24 months: 12.4%, 12.5%, 12.5%, 12.5% **3.** 21.8%

EXERCISE 4·9, Page 119

2. $3214.29 **3.** (a) $8000 **4.** $105.17

EXERCISE 4·10, Pages 121-122

1. $16.36, $22.75, $28.75

EXERCISE 4·11, Pages 126-127

1. (a) $60 (b) $1500 (c) $5000 **2.** (a) $190 (b) (i) $10,000 (ii) $10,000 (iii) $10,000
3. (a) $156.25 (b) (i) $25,000 (ii) $25,000 (iii) 0 (c) $130.21 (e) $4,687.50 (f) $48,000
4. (a) $83.00 (c) $41.67 **5.** (a) $112.50 (b) $110.50 (c) $112.20 **6.** (a) $281.25 (b) $4,218.75 **7.** (a) $280.50 (c) $5,000 (d) $47.28

EXERCISE 4·12, Page 129

1. $198.58 **2.** $68.72 **3.** $77.42 **4.** $8.70 **5.** $12.35 **7.** $236.31 **9.** $129.30

EXERCISE 4·13, Page 132

2. $535.75 **3.** $78.75 **4.** $165.00 **5.** $294.05

REVIEW EXERCISE 4, Page 132

1. (a) 13.3% (b) 22.9%

CHAPTER 5

EXERCISE 5·1, Page 134

2. $\frac{16.2}{0.54}, \frac{162}{5.4}, \frac{810}{27}, \frac{8100}{270}, \frac{16,200}{540}$ **3.** $\frac{30}{1}$

4. 16.2:0.54, 162:5.4, 810:27, 8100:270, 16,200:540

EXERCISE 5·2, Page 135

1. 48:1:22 **2.** 10:20:40 **3.** 6500:10,000:8000 not including the brother.

EXERCISE 5·3, Page 136

1. $k = 4$ **2.** $k = 5$ **3.** $k = 2.5$ **6.** $k = \frac{1}{14}$ **10.** $k = \frac{1}{6}$ **12.** $k = \frac{1}{16}$ **13.** $k = 4$ **18.** yes **19.** no **20.** yes

EXERCISE 5·4, Page 137

1. yes **2.** no **3.** no **4.** no **5.** no **6.** no **7.** 18 **8.** 20 **9.** 88
10. 2 **11.** 9 **12.** 12 **13.** 4 **14.** $\frac{12}{7}$ **15.** 29.9 **16.** 96 **17.** 20 **18.** 16

EXERCISE 5·5, Page 138

1. 7.2, 12.0, 4.8 **2.** 90, 60, 300 watt-hours respectively **3.** $66,600, $44,400, $11,100

EXERCISE 5·6, Page 138

1. 21 **2.** 20 **3.** 7 **4.** 6 **5.** 10.1 **6.** 5.19 **7.** 18 **8.** 100 **9.** 74.2

EXERCISE 5·7, Pages 139-141

1. 460 mi **6.** 0.52 mi **7.** 67.2 cm, 76.8 cm, 115.2 cm, 35.2 cm **8.** 13,200:1850, 21.4 ft **9.** 1.03 mi
10. 900% **11.** 0.09 lb, 27,800,000 gal **12.** (a) 0.05 lb (b) 0.45 lb **13.** 8.74g
14. (a) 14.3% (b) 12,100,000 lb **15.** 20, $33\frac{1}{3}$, $97\frac{2}{9}$, 28, 14.8, 80, $66\frac{2}{3}$, 40, 12.5
16. 3 tons **17.** 7,520,000 **18.** 3,760,000,000 kg **19.** 8,200,000 cars, 4,100,000,000 kg

REVIEW EXERCISE 5, Page 144

1. (a) yes (b) yes (c) yes (d) no **2.** 1:4 **3.** 1:8 **4.** $172,500, $129,375, $43,125
5. 97.7 ft **6.** $21\frac{1}{3}$ ft **8.** $2,420

CHAPTER 6

EXERCISE 6·2, Page 149

1. (a) $7,051.20 (b) $3,525.60 (c) $3,254.40 (d) $1,627.20 **2.** (a) $163.46, $326.92, $354.17, $708.33

EXERCISE 6·3, Page 151

1. $68.20 **2.** $74.36 **3.** $93.80 **4.** $303.30 **5.** $116.50 **6.** (a) $13.60 (b) $31.36 (c) $6.72 (d) $24.00 (e) $18.96 (f) $18.24 **7.** (a) $5.40 (b) $27.80 (c) $11.68 (d) $37.80

EXERCISE 6·5, Pages 156-157

1. $12.25, $18.90, $11.10, $0.35 **2.** $81.94, $73.43, $86.18, $52.82 **3.** (a) $1.37 (b) $0.20 (c) $1.53 **4.** Ken Jones: $1.89, $20.10, $94.51 Miss Jordan: $0.97, $8.05, $56.48 Frank Scarlett: $1.60, $11.65, $87.25, Douglas Faire: $0.24, 0, $24.76

CHAPTER 7

EXERCISE 7·1, Page 172

1. (a) four on the other hand (b) four on the foot (c) three on the other foot (d) both hands on the other Indian

EXERCISE 7·2, Page 163

1. (a) 202,229 (b) 3,000,507 (c) 30,030 (d) 4254 (e) 320,208
3. (a) 23 (b) 52 (c) 94 (d) 72,194 (e) 1394

EXERCISE 7·3, Page 165

1. (a) 366 (b) 251 (c) 140 (d) 3453 (e) 7080 (f) 737

EXERCISE 7·4, Page 166

1. (a) XLIII (b) XLIX (c) XCIV (d) XCVIII (e) XCIX (f) CDXL (g) CDXC (h) CDXCIX (i) CMXLV (j) CMXCIII (k) CMLXXXVII (l) CMXCIX **2.** (a) 27 (b) 246 (c) 578 (d) 1970

EXERCISE 7·6, Pages 168-169

2. 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, 10000, 10001, 10010, 10011, 10100 **3.** 1, 10, 100, 1000, 10000 **4.** 1, 2, 4, 8, 16

EXERCISE 7·7, Pages 169-170

1. (a) 3 (b) 13 (c) 11 (d) 17 (e) 27 (f) 23 (g) 24 (h) 30
2. (a) 1111 (b) 110 (c) 1100 (d) 11111 (e) 11100 (f) 10010 (g) 1010 (h) 11010

EXERCISE 7·8, Page 171

1. (a) 220 (b) 188 (c) 199 (d) 248 (e) 128 (f) 170 (g) 136 (h) 255
2. (a) 11111010 (b) 10011111 (c) 10101111 (d) 10001110 (e) 10010111 (f) 10101001 (g) 10000010 (h) 11011110 (i) 11001000

EXERCISE 7·9, Page 172

1. (a) 153 (b) 222 (c) 79 (d) 330 (e) 151 (f) 113 (g) 263 (h) 190 (i) 3743
2. (a) 173 (b) 6216 (c) 8063 **3.** (a) 130 (b) 111 (c) 3402

EXERCISE 7·11, Page 174

1. (a) 0100 1001 0001 (b) 0010 0110 0101 0100 (c) 0011 0111 1000 0001 0000 (d) 0111 0100 0110 (e) 0010 0100 0001 0011 (f) 0100 0101 0110 0111 0100 0011 0010
2. (a) 804 (b) 9167 (c) 982 (d) 1000 (e) 999 (f) 49235
3. (e) 96808421 (f) 863059800025 (g) 219974861 (h) 873642

EXERCISE 7·12, Page 175

2. (a) K (b) S (c) H (d) U (e) W (f) F **5.** HELP ME

REVIEW EXERCISE 7, Page 176

1. (a) 86 (b) 147 (c) 83 (d) 140 **2.** (a) 110110100 (b) 3221 (c) 664 **3.** (a) 111000 (b) 1300 (c) 20000 **4.** (a) 0010 0011 (b) 1000 1001 (c) 0010 0100 0111 0110
5. (a) 40 (b) 601 (c) 175
6. ALGEBRA **7.** 0100 0100 0101 1000 0101 0000 0100 1000 0101 0010 0101 0100

CHAPTER 8

EXERCISE 8·1, Pages 179-180

3. (a) $32a + 11b$ (b) $80b + 72a$ (c) $30x + 42y$ (d) $46w + 38v$ (e) $154t + 155s$ **4.** (a) $18a^2$ (b) $208a^2$ (c) $156b^2$ (d) $703x^2$ (e) $36a^2 + 28ab$ (f) $30mn + 10m^2$ (g) $255vw + 1173w^2$ (h) $2184vz + 1638wz$
5. (a) $7a + 4b$ (c) $11a - 12b - 32c$ (c) $-a - 16c - 13b$ (d) $27a + 2b$
(e) $16a - 12b$ (f) $a + 3b - 7c$ (g) $13a + 15b - 6c$ (h) $4a + 2b - 2c$
6. (a) $20a = 40$ ft, $18a = 36$ ft, $12b = 36$ ft, $22b = 66$ ft (b) 6456 sq. ft.

7. (a) $3a - 4b$ (b) $6x - 3y$ (c) $12w - 9v$ (d) $2a - 4$ (e) $5a - 3$ (f) $\frac{3}{2}a$
(g) $\frac{14}{3} - 7m$ (h) $\frac{1}{2} + \frac{13}{18}y$ (i) $\frac{11}{4}a^2 - 2a + 1$ (j) $4w - 3 - \frac{2}{w}$

8. (a) 0 (b) $-16a + 10$ **9.** $84a - 40b$, $630ab - 525b^2$; $96a + 84b$, $495a^2 + 1386ab$; $34m + 70n$, $146mn + 276n^2$

EXERCISE 8·2, Page 182

1. $a^2 + 8ab + 15b^2$ **2.** $a^2 + 11ab + 30b^2$ **3.** $2a^2 + 9ab - 18b^2$ **4.** $12a^2 - ab - 63b^2$
5. $24x^2 - 22xy - 30y^2$ **6.** $42x^2 - 9xy - 6y^2$ **7.** $12m^2 - 66mn + 30n^2$ **8.** $42m^2 - 74mn + 24n^2$
9. $a^2 + 7ab + 10b^2$ **10.** $a^2 + 13ac + 42c^2$ **11.** $x^2 + 3xy - 10y^2$ **12.** $x^2 + 2xy - 35y^2$
13. $8m^2 + 2mn - 15n^2$ **14.** $48p^2 - 72pr - 48r^2$ **15.** $72p^2 - 78pr + 18r^2$ **16.** $40p^2 - 71pr + 21r$
17. $\frac{1}{8}p^2 - \frac{3}{16}pr - \frac{1}{8}r^2$ **18.** $\frac{3}{8}p^2 - \frac{5}{16}pr - \frac{3}{8}r^2$ **19.** $\frac{3}{4}x^2 - \frac{5}{6}xy + \frac{2}{9}y^2$ **20.** $2x^2 - \frac{17}{36}xy - 2y^2$

21. $10x^2 - 21xy + 9y^2$, $9x^2 + 3xy - 2y^2$, $9x^2 - 6xy + y^2$

EXERCISE 8·3, Pages 183-184

1. (a) $(t + n)$ ft (b) $(p + r)$ ft (c) $(y - z)$ ft **2.** (a) $x(a - b)$ (b) $m(s - t - r)$
(c) $t(r - s + p)$ (d) $t(x - y)$ (e) $t(z - x + v)$ (f) $n(m - t + s)$ (g) $\frac{1}{2}h(r - t)$
(h) $\frac{2}{3}R(1 - R)$ **3.** (a) $4(2 + 3a + 4a^2)$ (b) $6abc(b - 2c)$ (c) $7xy(3y - 7x)$
(d) $\pi(r^2 - R)$ (e) $9at(7a - 5t + 8)$ (f) $\frac{1}{2}R(R - 3T)$
4. (a) $6x(x - 9y)$ (b) $(x + 2y)(3b + 2c)$ (c) $(2a - 3c)(x + y)$ (d) $(5a - 7d)(y^2 - x^2)$ (e) $(T^2 - t)(k + K)$
5. 84π sq. ft., 81π sq. ft., 115π sq. ft., 128π sq. ft. **6.** $\frac{1}{2}b(H - h)$, $22\frac{1}{2}$ sq. ft., 143 sq. ft.
7. 12π cc **8.** 438.75π cu. in. **9.** 1890 cc

EXERCISE 8·4, Page 185

1. $(x + y)^2$, $(x + 2y)^2$, $(2x + 3y)^2$ **2.** (a) $d^2 + 2dc + c^2$ (b) $s^2 + 4st + 4t^2$ (c) $s^2 - 4st + 4t^2$
(d) $9r^2 + 12rt + 4t^2$ (e) $9r^2 - 12rt + 4t^2$ (f) $16x^2 + 24xy + 9y^2$

EXERCISE 8·5, Pages 186-187

1. $3s + 2r$, $2t + n$ **2.** (a) $(2x - 1)^2$ (b) $(3x + 5)^2$
(d) $(3m - 5n)^2$ (e) $(6a - 7b)^2$ (f) $(5a - 3b)(5a - 12b)$ (g) $(9k - 5l)^2$ (h) $(7m - 4n)^2$
3. (a) $20mn$ (b) $2mn$ (c) $24xy$ (d) $84kl$ (e) $24xy$ (f) $2mn$ (g) y^2 (h) y^2
(i) $16a^2$ (j) $4w^2$ (k) n^2 (l) n^2 (m) l^2 (b) $36n^2$

EXERCISE 8·6, Pages 188-189

13. 2496 **14.** 1599 **15.** 9999 **16.** 4899 **17.** 999,999 **18.** 6384 **20.** $(c - d)(c + d)$
21. $(m - n)(m + n)$ **22.** $(6y - 5x)(6y + 5x)$
23. $(10s - 9d)(10s + 9d)$ **25.** $(\frac{1}{2}x - \frac{1}{3}y)(\frac{1}{2}x + \frac{1}{3}y)$ **26.** $(\frac{3}{2}y - \frac{5}{7}x)(\frac{3}{2}y + \frac{5}{7}x)$
27. $(\frac{1}{3}x - \frac{1}{4}y)(\frac{1}{3}x + \frac{1}{4}y)$ **29.** $(R - 10)(R + 10)$ **30.** $(T - 11)(T + 11)$
31. $(1 - 8a)(1 + 8a)$ **32.** $(a^2 + b^2)(a + b)(a - b)$ **33.** $2(2y - 3x)(2y + 3x)$
34. $(c^2 + d^2)(c + d)(c - d)$ **35.** $(d^2 + 1)(d + 1)(d - 1)$ **36.** $4(w + 1)(w - 1)$
37. $8(x + 2y)(x - 2y)$ **39.** 119π sq. in., 27π sq. ft. **40.** 715.62π cu. in.
41. 52π cu. ft.

REVIEW EXERCISE 8, Page 190

1. (a) $3a + 5b$ (b) $-8m - 31n$ **2.** (a) $60k^2 - 35kl$ (b) $4a^2 - 12am + 9m^2$ (c) $16a^3 - 32a^2 - 40a$ (d) $4a$
(e) $7a^2 - 3$ (f) $\frac{15}{2}x^2 - 6x - 1$ **3.** (a) $15a^2 - 28ab + 12b^2$ (b) $14x^2 + 17xy - 6y^2$
(c) $9a^2 - 30ab + 25b^2$ (d) $4x^2 - 9y^2$ **4.** (a) $3mn(m - 2n)$ (b) $(2v - w)^2$ (c) $(3m + 5n)^2$
(d) $(8 - 9y)(8 + 9y)$ **5.** 2496 **6.** 16π sq. in.

CHAPTER 9

EXERCISE 9·1, Page 194

1. $\triangle ABC \equiv \triangle EDF$, $\triangle BAC \equiv \triangle DEF$, $\triangle CBA \equiv \triangle FDE$

EXERCISE 9·3, Pages 198-200

1. (a) S.S.S. (b) S.A.S. (d) A.A.S. (f) S.S.S. (h) S.A.S. **2.** (a) A.A.S. (b) S.S.S.
(c) S.A.S. **5.** (a) 14, 10 (b) 4 (c) negative lengths not allowed.

EXERCISE 9·5, Page 207

1. (a) $a = 60°$, $b = c = 120°$ (b) $d = 110°$, $e = f = 70°$ (c) $g = 114°$, $h = i = 66°$
(d) $f = b = 70°$, $a = e = g = 110°$ (e) $a = b = c = d = e = 90°$ (f) $a = f = 40°$,
$b = g = 140°$, $c = 70°$, $d = 110°$, $e = 70°$

EXERCISE 9·6, Page 208

2. (a) True statement. If I live in Canada, then I live in Winnipeg. not true.
(b) Not necessarily true. If Mary has wet hair, she went swimming. not true.
(c) Not true. If Mr. Trudeau is Liberal leader, he is Prime Minister. not true.
(d) True. If $y = x$, then $x = y$. true. (e) True. If $\angle B$ is a right angle, then $m \angle B = 90°$. true.

EXERCISE 9·7, Page 210

1. 78, 39.5, 1024, 1204, 58.9 **2.** 110 **3.** 652.8, 1254.4, 1920

EXERCISE 9·8, Page 213

2. 110°, 110°, 14 in.

EXERCISE 9·9, Pages 216-217

1. a, d **2.** (a) 556 ft (b) yes (c) 7.25 min, 5.94 min **3.** 30.4 ft **4.** 16,900 sq. ft.
5. (a) 127 ft (b) 108 ft **7.** (a) 6.25 ft, 11.8 ft, 5.3 ft, 11.4 ft, 20.9 ft (b) 139.4 ft
8. (a) 520 (b) 575 (c) 6610 (d) 7230 (e) 803 (f)1140(g) 8.6% (h)29.6%(j) 614, 690
9. 6120 **10.** (a) 4540 (b) $8\frac{1}{3}\%$, 4950

REVIEW EXERCISE 9, Page 226

1. not necessarily congruent, congruent (S.S.S.), congruent (S.A.S.), not necessarily congruent
2. no **3.** 80° **5.** 33°, 54°, 93° **6.** 151.2 cu. in. **7.** 154.16 sq. units **9.**39° **10.** 220 cu. in.
12. If the opposite sides of a quadrilateral are parallel, then the quadrilateral is a parallelogram.

CHAPTER 10

EXERCISE 10·1, Page 229

1. (a) 27.1 (b) 15.5 (c) 58.9 (d) 44.8 **2.** 42.7 ft

EXERCISE 10·2, Page 231

1. 4.7 ft **2.** 18.8 ft **3.** 5 ft **4.** 4 ft **5.** 9.4 mi **6.** 7.05 ft, 7.98 in, 300 ft, 658 mi

EXERCISE 10·3, Pages 232-233

1. 9.4, 18.8, 26.8, 27.0, 61.0, 34.6, 31.5, 13.9, 40.4, 21.8 **2.** 1.5 ft, 20 yd, 5.77 in, 5.02 mi, 29.6 ft, 9.4 yd, 39 ft **3.** 7.5 ft **4.** (a) 19.1 mi (b) 11 mi

EXERCISE 10·4, Page 236

1. (a) 0.8387 (b) 0.7314 (c) 0.5592 (d) 0.9511 (e) 0.9962 (f) 0.8660 (g) 0.1392 (h) 0.9272 (i) 0.2079
2. $y = 67.14$, $x = 126.3$ **3.** $y = 61.67$, $x = 26.18$ **4.** $y = 29.91$, $x = 39.69$
5. $y = 484.1$, $x = 326.6$ **6.** (a) 9 (b) 63 (c) 29 (d) 24 (e) 76 (f) 72

EXERCISE 10·5, Pages 240-241

1. $\{x \in R \mid 0 \leq x \leq 1\}$ **2.** $\{x \in R \mid 0 \leqslant x \leqslant 1\}$ **3.** 0, 57.29, no **4.** sometimes, never **5.** sometimes, never
6. sometimes, sometimes **7.** $\frac{1}{3}$, 18.4° **8.** 4.15, 76.4° **9.** 18.6, 51.5, 80.5, 46, 7.46, 7.65
10. (a) 26.0° (b) 45.5° (c) 72.6° **11.** (a) 64.0° (b) 44.5° (c) 17.4°
12. (a) 36.9° (b) 36.9° (c) 33.6° (d) 41.8° **13.** 61 ft **14.** 226 ft **15.** 208 ft

REVIEW EXERCISE 10, Page 250

1. 0.4540 **2.** (a) 36.2° (b) 36.8° **3.** 28.6° **4.** 14.3 ft **5.** 587.5 in

CHAPTER 11

EXERCISE 11·1, Page 254

1. vectors (b), (d), (e), (g) **4.** *A*, N68°E, 15; *B*, N10°W, $12\frac{1}{2}$; *C*, W15°S, $11\frac{1}{4}$; *D*, S32°E, $7\frac{1}{2}$

EXERCISE 11·2, Page 257

1. (a) 11 miles east (b) 1 mile south (c) 10 pounds upward (d) 140 mph north
(e) 720 miles N33.7°E (f) 106 pounds N48.8°E **3.** 80.6 naut. miles

EXERCISE 11·3, Pages 259-260

1. 33.6 mph 63.4° from shoreline **2.** 177 mph S48°W **3.** 705 mph N56°E **4.** 6.8 knots N80°E

EXERCISE 11·4, Page 261

1. (a) 540 mph N40°E (b) 936 ft S34°E (c) 1944 ft/sec/sec N (d) 400 pounds straight ahead
(e) 1764 pounds downward

EXERCISE 11·5, Page 262

1. 8.05 blocks E29.8°S **2.** 77.5 mi S57°W **3.** 3.24 mi N19°E

EXERCISE 11·6, Pages 266-267

1. 77.6 g upward **2.** 126 pounds S10°W **3.** 600 pounds **4.** 18°, 3.6 hrs **5.** 4.2 mi N14°W
6. 178 mph S55°E **8.** 0

REVIEW EXERCISE 11, Page 267

1. magnitude, direction **2.** (b) 93 pounds (c) N1°W (d) 93 pounds N1°W
3. 9.85 blocks E24°S **4.** 14.4 knots S25°E **5.** (a) 66 mph north (b) 54 mph north
(c) 60.3 mph N6°W

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